A note to the reader

During the preparation of this paper, I could not quite get rid of a nagging suspicion that such a very simple idea as underlies the accompanying proof of the Cauchy theorem must surely already exist in the literature? However, all my searching did not produce any results, so I went ahead and published.

As it turns out, my first intuition was entirely correct, as the argument has indeed been published before [*].

An intriguing question that arises from this is: Why was Výborný's proof seemingly universally ignored? One would have thought that the proof might find its way into some textbooks over the course of nearly 30 years, but apparently it has not. If the present paper helps to reverse that sorry state of affairs, I will consider it a success. (I admit that, though I can no longer claim an original proof, I am somewhat pleased with the exposition, as well as the historical notes.)

Please make sure that this note accompanies any copies of this paper, so that Výborný's contribution will be properly acknowledged if (when?) it ever gets into the textbook literature.

Trondheim, 17 October 2008
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Reference

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On Goursat's Proof of Cauchy's Integral Theorem

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1. INTRODUCTION. One standard proof for the Cauchy integral theorem goes something like this: First one proves it for triangular paths, and uses this to establish the existence of an antiderivative on star shaped regions. The Cauchy integral theorem follows on such regions. Next, the homotopy version of the theorem is derived from this, typically with some difficulty of a didactic nature.

The purpose of this note is to point out that the homotopy version is easily derived directly, by the simple expedient of employing Goursat's trick in the domain of the homotopy.

2. THE MAIN RESULT. For present purposes, a complex function f is called *analytic* in a region Ω if its derivative f' exists at every point in Ω. Thus by definition, whenever $z_0 \in \Omega$ then

$$f(z) = f(z_0) + f'(z_0) \cdot (z - z_0) + o(|z - z_0|), \qquad z \to z_0. \tag{1}$$

Note that we do *not* require f' to be continuous, or even locally bounded. That can be inferred later, as a consequence of Cauchy's integral formula, which in turn is derived from Cauchy's integral theorem. This derivation is quite standard, so I skip it here.

I skip the proof of the following special case of the integral theorem. It is easily derived directly from the definition of the integral.

Lemma 1. *If a and b are complex constants and γ is a closed, rectifiable path, then*

$$\int_\gamma (a + bz)\, dz = 0.$$

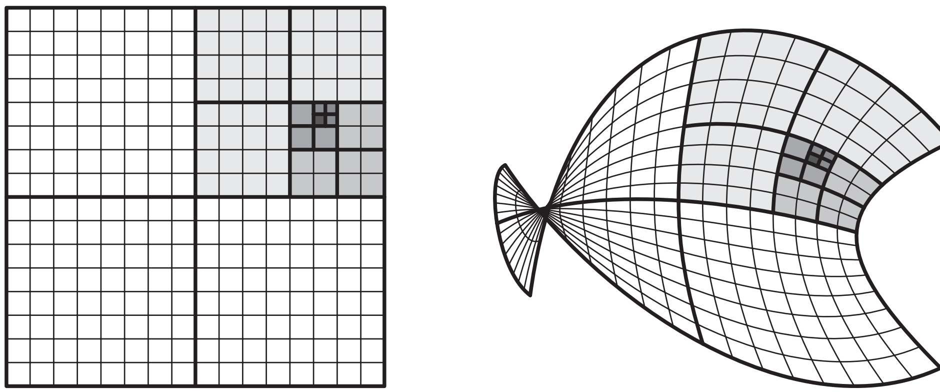

Figure 1. Left, the squares $Q_0 \supset Q_1 \supset Q_2 \supset Q_3 \supset Q_4$ shown in successively darker colors. Right, the image of these squares under the homotopy σ.

By a *homotopy* in Ω we shall mean a continuous mapping $\sigma\colon [0,1]\times[0,1] \to \Omega$. For technical reasons, we consider *Lipschitz continuous* homotopies. That is, we assume that there is a constant L so that

$$|\sigma(s,t) - \sigma(s^*,t^*)| \le L\sqrt{(s-s^*)^2 + (t-t^*)^2} \qquad \text{for } s, s^*, t, t^* \in [0,1].$$

The restriction to Lipschitz continuous homotopies is not a serious one.

A parametrized path α in $[0,1]\times[0,1]$ is mapped in a natural way to a path in Ω, by composing it with σ. We shall therefore write $\sigma\circ\alpha$ for this path. Of particular interest to us will be the *boundary* of $[0,1]\times[0,1]$. We parametrize it by arclength, starting at $(0,0)$ and proceeding around the boundary in the counterclockwise direction. We shall write α_0 for this path.

Theorem 2. *Let f be analytic in a region Ω in the complex plane. If $\sigma\colon [0,1]\times[0,1] \to \Omega$ is a Lipschitz continuous homotopy, then*

$$\int_{\sigma\circ\alpha_0} f(z)\,dz = 0.$$

Proof. Assuming this is not the case, we can rescale f and assume

$$\int_{\sigma\circ\alpha_0} f(z)\,dz = 1.$$

Now divide $Q_0 = [0,1]\times[0,1]$ into four squares of side $\frac{1}{2}$. The boundary curve of each smaller square, parametrized in direct analogy with α_0, gives rise to an integral, and the sum of these four integrals is the integral around $\sigma\circ\alpha_0$, i.e., 1. Thus one of these smaller squares, which we shall call Q_1, and whose boundary curve we shall call α_1, satisfies

$$\left|\int_{\sigma\circ\alpha_1} f(z)\,dz\right| \ge \frac{1}{4}.$$

Next we divide Q_1 into four smaller squares, one of whose boundary curves will produce an integral of absolute value at least $1/16$, and so forth.

In general, we get squares $Q_0 \supset Q_1 \supset Q_2 \supset \cdots$, where Q_n has sides of length 2^{-n}, and its boundary curve α_n satisfies

$$\left|\int_{\sigma\circ\alpha_n} f(z)\,dz\right| \geq 2^{-2n}. \tag{2}$$

Now there is a point $(s_0, t_0) \in [0, 1] \times [0, 1]$ common to all the squares Q_n. Write $z_0 = \sigma(s_0, t_0)$. Then $|z - z_0| \leq 2^{1-n}L$ for any z on the path $\sigma \circ \alpha_n$, where L is the Lipschitz constant of σ.

We now employ (1) and Lemma 1, concluding that only the final term in (1) will contribute to the integral of f around a closed contour, so that

$$\int_{\sigma\circ\alpha_n} f(z)\,dz = o(2^{-2n}), \qquad n \to \infty,$$

with a factor $O(2^{-n})$ coming from the length of $\sigma \circ \alpha_n$ (which is at most $2^{2-n}L$) and the other factor $o(2^{-n})$ coming from the final term of (1). But this contradicts (2), and the proof is complete. ■

The usual homotopy forms of the Cauchy integral theorem follow immediately from the theorem above. If γ_0 and γ_1 are two paths in Ω with $\gamma_0(0) = \gamma_1(0) = \alpha$ and $\gamma_0(1) = \gamma_1(1) = \beta$, and these paths are homotopic with fixed end points, this means we can find a homotopy σ with $\gamma_j(t) = \sigma(j, t)$ for $j = 0, 1$, while $\sigma(s, 0) = \alpha$ and $\sigma(s, 1) = \beta$ for all s. But then $\sigma \circ \alpha_0$ is composed of γ_1 followed by the reverse of γ_0, with a constant path inserted in front of each. So the conclusion of the theorem is just the identity $\int_{\gamma_1} f(z)\,dz - \int_{\gamma_0} f(z)\,dz = 0$.

Similarly, if the two paths are closed and homotopic via closed paths, i.e., $\sigma(s, 0) = \sigma(s, 1)$ for all s, we find that $\sigma \circ \alpha_0$ is composed of four paths: $s \mapsto \sigma(s, 0)$ followed by γ_1 and then the reverse of the two paths $s \mapsto \sigma(s, 1)$ and γ_0. But since the first and third of these are each other's reverses, the corresponding integrals will cancel, and again we are left with $\int_{\gamma_1} f(z)\,dz - \int_{\gamma_0} f(z)\,dz = 0$.

3. NON-LIPSCHITZ HOMOTOPIES. It remains to employ a simple approximation argument to show that our restriction to Lipschitz continuous homotopies is harmless.

First, consider the approximation of paths. If $\gamma\colon [0, 1] \to \Omega$ is a rectifiable path and $0 = t_0 < t_1 < t_2 < \cdots < t_n = 1$ is a partition of [0, 1], we can create a new path $\tilde{\gamma}$ by requiring that $\tilde{\gamma}(t_k) = \gamma(t_k)$ for $k = 0, 1, \ldots, n$ and interpolating linearly in between. Proving that

$$\int_{\tilde{\gamma}} f(z)\,dz \to \int_{\gamma} f(z)\,dz$$

as the partition is refined is quite straightforward, using the uniform continuity of f in a compact neighbourhood of γ and the finite length of γ.

Consider next an arbitrary homotopy $\kappa\colon [0, 1] \times [0, 1] \to \Omega$. Given any partition $0 = t_0 < t_1 < t_2 < \cdots < t_n = 1$ of [0, 1], create a Lipschitz continuous homotopy σ by requiring that $\sigma(t_j, t_k) = \kappa(t_j, t_k)$ for $j, k \in \{0, 1, \ldots, n\}$ and interpolating bilinearly in each subrectangle:

$$\begin{aligned}\sigma\big(ut_j + (1-u)t_{j-1}, vt_k + (1-v)t_{k-1}\big)\\ &= (1-u)(1-v)\kappa(t_{j-1}, t_{k-1}) + u(1-v)\kappa(t_j, t_{k-1})\\ &\quad + (1-u)v\kappa(t_{j-1}, t_k) + uv\kappa(t_j, t_k).\end{aligned}$$

for $u, v \in [0, 1]$ and $j, k \in \{1, \dots, n\}$. If the partition is fine enough, σ will approximate κ uniformly and hence map $[0, 1] \times [0, 1]$ into Ω, so we can apply the theorem to σ. Also, the path $\sigma \circ \alpha_0$ is a piecewise linear approximation to $\kappa \circ \alpha_0$, so we conclude that $\int_{\kappa\circ\alpha_0} f(z)\,dz = 0$. Of course, this approximation procedure requires that $\kappa \circ \alpha_0$ be rectifiable, but no such assumption beside continuity is needed concerning κ in the interior of $[0, 1] \times [0, 1]$.

The case of a homotopy between closed paths is slightly different: then we wish to avoid any assumption on the rectifiability of the path $s \mapsto \kappa(s, 0) = \kappa(s, 1)$. But in this case, the approximation σ will also satisfy $\sigma(s, 0) = \sigma(s, 1)$, so the corresponding integrals will cancel each other. Thus we only need to consider the other two sides of the square, and the desired result follows.

4. HISTORICAL REMARKS. Cauchy first communicated the integral theorem to the Paris Academy on 22 August 1814, as part of a memoir concerned with other matters [**2**]; albeit without the explicit presence of a complex variable. The first general form of the theorem was communicated to the Academy in 1825, and was printed as a brochure [**4**]. An extract of this brochure appeared in [**3**]. More details of the story can be found in Kline's book [**8**, pp. 634–638]—but note that this was written before the final volume of Cauhy's *Œvres complètes* was published, so Kline may well have been unaware of [**4**].

Goursat presented a proof [**6**] in 1884 for the case of a simple closed curve, by dividing up the interior into small squares, showing that the integral around each square of side l is bounded by εl^2, and then adding the results. To obtain this estimate for arbitrary $\varepsilon > 0$ he needs the uniform continuity of f'. In his 1900 paper [**7**], Goursat finally dispensed with the *a priori* assumption of continuity of f'. The argument is as before, but he now subdivides each square until the necessary estimate holds for each of them; i.e.,

$$|f(z) - f(z_0) - f'(z_0)\cdot(z - z_0)| < \varepsilon|z - z_0| \tag{3}$$

for every z on the boundary of the subsquare, where z_0 is some fixed point in the subsquare. The important point here is that the same ε will do for each subsquare.

He proves that he only needs a finite number of subdivisions by noticing that otherwise, one gets an infinite descent of squares violating (3), and an application of (1) at the limit point results in a contradiction. But after noting this, he adds the integral estimates for individual squares as before.

Moore [**9**] cleaned up the proof a bit, paying more careful attention to the treatment of the boundary curve. He also introduced the current idea of proof by contradiction, by subdividing and always selecting a part where the desired conclusion is maximally violated, and then applying the definition of the derivative at the resulting limit point. (As Moore states in a footnote, the proof "may easily be cast in the direct form", which observation he credits to Maschke.)

Pringsheim presented a more severe criticism [**10**] of Goursat's treatment of the boundary curve in 1901. He pointed out that these problems disappear if the proof technique is applied to a simple geometric figure such as a triangle. Then the theorem follows for simple polygonal paths in a simply connected domain by triangulating the interior of the path, and finally one gets it for a general path by approximating it by polygonal paths.

5. CONCLUDING REMARKS. I cannot end this account without directing the reader's attention to Dixon's remarkably short and elegant proof of the global version of the Cauchy theorem [**5**]. It only requires the basics of complex function theory in convex sets. (A variant of this proof, due to Beardon [**1**], requires even less of analytic function theory, albeit at a slight cost in complexity.) To a large degree this proof obviates the need for a homotopy version of Cauchy's theorem. Indeed, Dixon states:

> It is reasonable to argue that the concept of homotopy in connection with Cauchy's theorem is as extraneous as the notion of Jordan curve.

Be that as it may, not many textbooks of complex analysis seem to have taken his advice to heart, and most still follow the homotopy route. Personally, I think Dixon has a point, but it somehow feels artificial to first develop complex function theory in star shaped or convex regions only, in order to bring in heavy artillery later and demolish the perceived need for this simplifying assumption. The present proof may be short and direct enough to replace that first phase of theory building. The powerful magic of Dixon's proof can (and should) still be brought to bear later.

ACKNOWLEDGEMENTS. I would like to thank Peter Lindqvist for enlightening discussions on the historical origin of Cauchy's integral theorem. Also, my thanks to a referee for directing my attention to Beardon's book [**1**].

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