The double (and triple, and ...) mean value theorem

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A colleague came across some mentions of the double and triple mean value theorems today (see the references). As neither he nor I had heard about these theorems before, we were naturally intrigued.

All the references seem to state the double mean value theorem in roughly the same form: *Let* $f \in C^2(\mathbb{R})$*, and* $\max\{|\eta|, |\lambda|\} \ll \xi$*, then*

$$|f(\xi+\eta+\lambda) - f(\xi+\eta) - f(\xi+\lambda) + f(\xi)| \lesssim |f''(\theta)||\eta||\lambda|,$$

where $|\theta| \sim |\xi|$.

We were actually somewhat at a loss as to the exact meaning of this statement, so I sat down and thought about it for a moment. No references were given to this – presumably – elementary result. Well, as it turns out, it *is* elementary, but perhaps not quite so obvious.

The ordinary mean value theorem can be deduced from a simple integration: If $f \in C^1(\mathbb{R})$ then

$$f(\xi+\eta) - f(\xi) = \eta \int_0^1 f'(\xi + t\eta)\,\mathrm{d}t. \tag{1}$$

We could easily leave it at that, or we could notice that the integral is a mean value of the continuous function f' over the interval with end points ξ and $\xi+\eta$, so it equals $f'(\theta)$ for some θ in that interval.

The double mean value theorem is easily deduced by two applications of the above. Assuming $f \in C^2(\mathbb{R})$, apply (1) to the function F given by $F(\xi) = f(\xi+\lambda) - f(\xi) = \lambda\int_0^1 f'(\xi+s\lambda)\,\mathrm{d}s$. Then $F'(\xi) = \lambda\int_0^1 f''(\xi+s\lambda)\,\mathrm{d}s$, and so (1) becomes

$$f(\xi+\eta+\lambda) - f(\xi+\eta) - f(\xi+\lambda) + f(\xi) = \eta\lambda \int_0^1\int_0^1 f''(\xi+s\lambda+t\eta)\,\mathrm{d}s\,\mathrm{d}t. \tag{2}$$

Again, we can rewrite the righthand side as $\eta\lambda f''(\theta)$ for some θ in the smallest interval containing all the function arguments on the lefthand side.

From this, any competent mathematician should be able to derive an estimate of the form used in the referenced papers. Moreover, this makes it abundantly clear precisely what is required to use the estimates.

The triple, and quadruple, etc., mean value theorems are all easily derived by repeating the same procedure. E.g., to get the triple mean value theorem, let $F(\xi)$ be the two sides of (2), but with η replaced by (say) γ, then apply (1) to this function. The result is

$$f(\xi+\eta+\lambda+\gamma) - f(\xi+\eta+\gamma) - f(\xi+\lambda+\gamma) - f(\xi+\eta+\lambda) + f(\xi+\gamma) + f(\xi+\eta) + f(\xi+\lambda) - f(\xi)$$
$$= \eta\lambda\gamma \int_0^1\int_0^1\int_0^1 f'''(\xi+s\lambda+t\eta+u\gamma)\,\mathrm{d}s\,\mathrm{d}t\,\mathrm{d}u.$$

The pattern and proof of the higher order mean value theorems should be obvious from this.

References

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[2] J. Colliander, M. Keel, G. Staffilani, H. Takaoka, T. Tao: *Sharp global well-posedness for KdV and modified KdV on* $\mathbb{R}$ *and* $\mathbb{T}$. J. Amer. Math. Soc. **16** (2003), 705–749. doi:10.1090/S0894-0347-03-00421-1

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