The two-dimensional isodiametric inequality

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The *isodiametric inequality* states that, of all bodies of a given diameter, the sphere has the greatest volume. A proof can be found, e.g., in Lawrence C. Evans & Ronald F Gariepy: *Measure theory and fine properties of functions*. This note is about a particularly simple and beautiful proof of the isodiametric inequality in two dimensions.

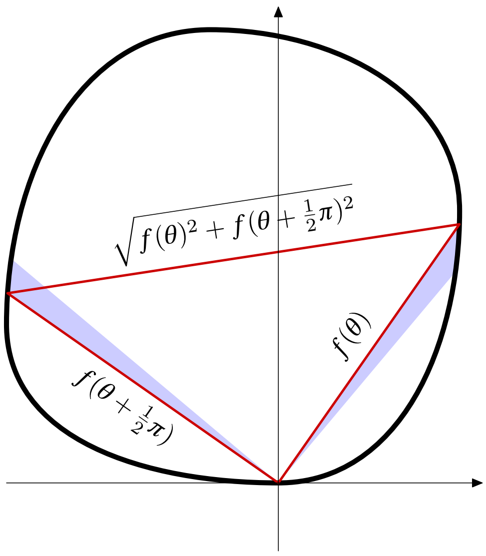
I originally posted this as a plea for help to discover the origin of the proof. I have since learned that it is found in *Littlewood's miscellany*, on p. 32.

The *diameter* of a body is defined as the supremum of the distances between two points in the body. Clearly, we need only show the isodiametric inequality for convex bodies, since taking the convex closure does not increase the diameter, nor does it decrease the area. Next, we may move the (convex) body so that it lies in the upper half plane, with the origin at its boundary. Thus we may describe the body in polar coordinates by

$$r \le f(\theta), \qquad 0 \le \theta \le \pi.$$

We now apply a bit of first year calculus to write the area of the body as

$$A = \tfrac{1}{2}\int_0^\pi f(\theta)^2\, d\theta.$$



We now split the integral in two halves and change the variable in the second half so that both halves can be written as the single integral

$$A = \tfrac{1}{2}\int_0^{\pi/2} \bigl(f(\theta)^2 + f(\theta+\tfrac{1}{2}\pi)^2\bigr)\, d\theta.$$

But here we recognize the integrand as the squared hypothenuse of the right triangle in the figure. By definition, the hypothenuse cannot be greater than the diameter d of the region:

$$f(\theta)^2 + f(\theta+\tfrac{1}{2}\pi)^2 \le d^2.$$

Thus

$$A \le \pi(d/2)^2$$

which is the isodiametric inequality, and so the proof is complete.