Differential forms and the polar planimeter

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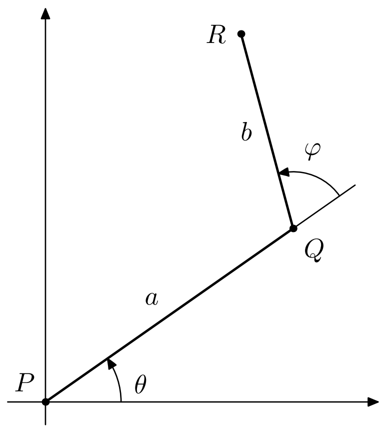

The *planimeter* is a mechanical device for measuring the area of a plane figure drawn on a sheet of paper. One common model consists of two interconnected arms of fixed lengths a and b, the first one free to pivot about a fixed point P. The connecting point Q rolls and slides on a wheel whose axis is parallel to the second arm. The user moves the end R of the second arm around the boundary of the region of interest, and aftwards reads off the resulting rotation of the wheel at Q. The total angle of rotation of this wheel is proportional to

$$\oint_C \cos\varphi \,\mathrm{d}\theta = \iint_R \sin\varphi \,\mathrm{d}\theta \wedge \mathrm{d}\varphi.$$

We shall show that this quantity is indeed proportional to the area within the curve C. For this purpose, it is convenient to use complex variables. The position of R can be written as

$$z = ae^{i\theta} + be^{i\psi} \qquad \text{where } \psi = \theta + \varphi.$$

We find

$$\begin{aligned}
\mathrm{d}x \wedge \mathrm{d}y &= -\tfrac{1}{4} i \,\mathrm{d}(z+\bar{z}) \wedge \mathrm{d}(z-\bar{z}) \\
&= -\tfrac{1}{2} i \,\mathrm{d}\bar{z} \wedge \mathrm{d}z = -\tfrac{1}{2} i (ae^{-i\theta}\,\mathrm{d}\theta + be^{-i\psi}\,\mathrm{d}\psi) \wedge (ae^{i\theta}\,\mathrm{d}\theta + be^{i\psi}\,\mathrm{d}\psi) \\
&= -\tfrac{1}{2} abi (e^{i(\psi-\theta)} - e^{i(\theta-\psi)}) \,\mathrm{d}\theta \wedge \mathrm{d}\psi \\
&= -ab \sin\varphi \,\mathrm{d}\theta \wedge \mathrm{d}\varphi
\end{aligned}$$

so that the area of interest is

$$\iint_R \mathrm{d}x \wedge \mathrm{d}y = -ab \iint_R \sin\varphi \,\mathrm{d}\theta \wedge \mathrm{d}\varphi,$$

and we have proved what we set out to do.

Remarks. It really should be possible to prove this using only elementary calculus including Green's theorem. But in practice, that seems remarkably hard to do. Hence this example shows the power of differential forms over more conventional methods, even in apparently very simple cases.

The above proof can just as easily be done using only real variables, albeit at the price of a little bit of trigonometry.

One can do this proof in a more intuitive geometric way by noting that the infinitesimal area swept out by the outer arm is $b\,\mathrm{d}m + \frac{1}{2} b\, d\psi$ where $\mathrm{d}m = a\cos\varphi\,\mathrm{d}\theta$ represents the sideways component of the outer arm's motion at Q. Here the point is that "$\mathrm{d}m$" is not an exact differential, while $\mathrm{d}\psi$ is, so the integral of the latter around a closed curve yields zero. My point in these notes, however, is to show how using differential forms produced the correct answer much more easily than a first-year calculus kind of application of Green's theorem would have done, and without the need for clever geometric reasoning – however much I enjoy such arguments.