The uniform boundedness theorem

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The purpose of this note is to present an alternative proof of the uniform boundedness theorem, without the need for the Baire category theorem.

I found this proof in Emmanuele DiBenedetto: *Real Analysis*. DiBenedetto refers to an article by W. F. Osgood: Nonuniform convergence and the integration of series term by term, *Amer. J. Math.*, **19**, 155–190 (1897). Indeed, the basic idea of the following proof seems to be present in that paper, although the setting considered there is much less general: It is concerned with sequences of functions on a real interval.

I have rewritten the proof a bit, splitting off the hardest bit into a lemma.

1 Lemma. *Let (X,d) be a complete, nonempty, metric space, and let F be a set of real, continuous functions on X. Assume that F is pointwise bounded from above, in the following sense: For any $x \in X$ there is some $c \in \mathbb{R}$ so that $f(x) \le c$ for all $f \in F$. Then F is uniformly bounded from above on some nonempty open subset $V \subseteq X$, in the sense that there is some $M \in \mathbb{R}$ so that $f(x) \le M$ for all $f \in F$ and all $x \in V$.*

Proof: Assume, on the contrary, that no such open subset exists.

That is, for every nonempty open subset $V \subseteq X$ and every $M \in \mathbb{R}$, there exists some $f \in F$ and $x \in V$ with $f(x) > M$.

In particular (starting with $V = X$), there exists some $f_1 \in F$ and $x_1 \in X$ with $f_1(x_1) > 1$. Because f_1 is continuous, there exists some $\varepsilon_1 > 0$ so that $f_1(z) \ge 1$ for all $z \in \overline{B_{\varepsilon_1}(x_1)}$.

We proceed by induction. For $k = 2, 3, \ldots$, find some $f_k \in F$ and $x_k \in B_{\varepsilon_{k-1}}(x_{k-1})$ so that $f_k(x_k) > k$. Again, since f_k is continuous, we can find some $\varepsilon_k > 0$ so that $f_k(z) \ge k$ for all $z \in \overline{B_{\varepsilon_k}(x_k)}$. In addition, we require that $B_{\varepsilon_k}(x_k) \subseteq B_{\varepsilon_k}(x_k)$, and also $\varepsilon_k < k^{-1}$.

Now we have a descending sequence of nonempty closed subsets

$$X \supseteq \overline{B_{\varepsilon_1}(x_1)} \supseteq \overline{B_{\varepsilon_2}(x_2)} \supseteq \overline{B_{\varepsilon_3}(x_3)} \supseteq \cdots,$$

and the diameter of $B_{\varepsilon_k}(x_k)$ converges to zero as $k \to \infty$. Since X is complete, the intersection $\bigcap_k \overline{B_{\varepsilon_k}(x_k)}$ is nonempty; in fact, $(x_k)_k$ is a Cauchy sequence converging to the single element x of this intersection.

But now $f_k(x) \ge k$ for every k, because $x \in \overline{B_{\varepsilon_k}(x_k)}$. However that contradicts the upper boundedness of F at x, and this contradiction completes the proof. ∎

2 Theorem. (Uniform boundedness) *Let X be a Banach space and Y a normed space. Let $\Phi \subseteq B(X,Y)$ be a set of bounded operators from X to Y which is pointwise bounded, in the sense that, for each $x \in X$ there is some $c \in \mathbb{R}$ so that $\|Tx\| \le c$ for all $T \in \Phi$. Then Φ is uniformly bounded: There is some constant C with $\|T\| \le C$ for all $T \in \Phi$.*

Proof: Apply Lemma 1 to the set of functions $x \mapsto \|Tx\|$ where $T \in \Phi$. Thus, there is an open set $V \subseteq X$ and a constant C so that $\|Tx\| \le C$ for all $T \in \Phi$ and all $x \in V$.

Pick some $z \in V$ and $\varepsilon > 0$ so that $\overline{B_\varepsilon(z)} \subseteq V$. Also fix $c \in \mathbb{R}$ with $\|Tx\| \le c$ whenever $T \in \Phi$. Now, if $\|x\| \le 1$ then $z + \varepsilon x \in V$, and so for any $T \in \Phi$ we get

$$\|Tx\| = \left\|\varepsilon^{-1}\bigl(T(z+\varepsilon x) - Tz\bigr)\right\| \le \varepsilon^{-1}\bigl(\|T(z+\varepsilon x)\| + \|Tz\|\bigr) \le \varepsilon^{-1}(M+c).$$

Thus $\|T\| \le \varepsilon^{-1}(M+c)$ for any $T \in \Phi$. ∎