The Riemann–Lebesgue lemma

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This little note is devoted to a proof of the Riemann–Lebesgue lemma (see p. 504 in the book). This proof is simpler, and the statement stronger, than in the book.

We actually need this greater generality in order to use it in Chernoff's proof of the Fourier representation theorem (see his *Monthly* paper, linked from my home page).[1]

We use the following notation for the nth Fourier coefficient of a 2π-periodic function f:

$$\hat{f}(n) = \frac{1}{2\pi}\int_{-\pi}^{\pi} f(x)e^{-inx}\,dx.$$

1 Lemma. (Riemann–Lebesque) *Assume that f is 2π-periodic, bounded and integrable. Then $\hat{f}(n) \to 0$ when $n \to \pm\infty$.*

Proof: We shall prove this only for real-valued functions. If f is complex-valued, the result will follow from the result applied to the real and imaginary parts of f separately.

First, we prove the result for an extremely special case: Namely, a *single step,* which is a function of the form

$$s(x) = \begin{cases} 1 & a+2k\pi \le x \le b+2k\pi, \quad k \in \mathbb{Z} \\ 0 & \text{otherwise} \end{cases}$$

where $a < b$ and $b - a < 2\pi$. Then

$$\hat{s}(n) = \frac{1}{2\pi}\int_a^b e^{-inx}\,dx = \frac{e^{-inb} - e^{-ina}}{2\pi i n} \to 0 \qquad \text{as } n \to \pm\infty$$

since the numerator is bounded and the denominator goes to infinity.

Second, since any step function is a linear combination of a finite number of single steps, the same result holds for step functions.

[1] There exists an even more general statement that is beyond us at this point: It requires the use of the Lebesgue integral, which is more general than the Riemann integral that is introduced in calculus.

Finally, now assume that f is integrable, and pick any $\varepsilon > 0$. It follows – practically direct from the definition of integrability – that there exists a step function s with

$$\frac{1}{2\pi}\int_{-\pi}^{\pi} |f(x) - s(x)|\,dx < \varepsilon.$$

From this we get

$$|\hat{f}(n) - \hat{s}(n)| = \frac{1}{2\pi}\left|\int_{-2\pi}^{\pi} \bigl(f(x) - s(x)\bigr)e^{-inx}\,dx\right| \le \frac{1}{2\pi}\int_{-2\pi}^{\pi} \bigl|f(x) - s(x)\bigr|\,dx < \varepsilon$$

as well. We have shown that $\hat{s}(n) \to 0$, so there is some N so that $|n| \ge N$ implies $|\hat{s}(n)| < \varepsilon$. Whenever $|n| \ge N$, then

$$|\hat{f}(n)| \le |\hat{f}(n) - \hat{s}(n)| + |\hat{s}(n)| < \varepsilon + \varepsilon = 2\varepsilon,$$

which finishes the proof. ∎

Notice that the Riemann–Lebesgue lemma says nothing about *how fast* $\hat{f}(n)$ goes to zero. With just a bit more of a regularity assumption on f, we can show that $\hat{f}(n)$ behaves roughly like $1/n$ or better. This is easy if f is continuous and piecewise smooth, as is seen from the identity $\widehat{f'}(n) = in\hat{f}(n)$, which arises from partial integration. Applying the Riemann–Lebesgue lemma to f' we conclude that $\hat{f}(n)$ is $1/n$ times something that goes to zero, so $\hat{f}(n) \to 0$ *faster* than $1/n$.

We can even drop the requirement of continuity: Just so long as f is piecewise smooth, partial integration yields a formula just like $\widehat{f'}(n) = in\hat{f}(n)$, with the addition of some extra terms coming from the points of discontinuity. But these extra terms are bounded, so this time we get $\hat{f}(n) \to 0$ *as fast* as $1/n$.

If f has more continuous derivatives, we can keep on differentiating: We get $\widehat{f^{(k)}}(n) = (in)^k\hat{f}(n)$, and conclude that $\hat{f}(n)$ goes to zero faster than n^{-k}.