The classification of isolated singularities

Harald Hanche-Olsen
hanche@math.ntnu.no

It seems to me that the book's treatment of isolated singularities is organized in a somewhat confusing fashion. I'll try to simplify.

Let f be a function which is analytic in a neighbourhood of some point z_0, *except* at the point z_0 itself. Then z_0 is called an *isolated singularity* of f.

Recall that under the stated assumption, f can be represented by its *Laurent series* at z_0:

$$f(z) = \sum_{n=-\infty}^{\infty} a_n (z-z_0)^n \qquad 0 < |z-z_0| < R$$

where the outer radius of convergence is a positive number (possibly infinite). This immediately leads to the following classification.

Removable singularity. z_0 is called a *removable singularity* of f if $a_n = 0$ for all $n < 0$.

In this case the series above is an ordinary power series, and if we were to (re)*define* $f(z_0) = a_0$ then the redefined function is in fact analytic at z_0. So the "singularity" has disappeared, which is why we called it removable.

To carry this a bit further (we shall need it later), let N be the smallest index n for which $a_n \neq 0$. (If there is none, then f is of course identically zero in a neighbourhood of z_0.) Then we can write

$$\begin{aligned} f(z) &= \sum_{n=N}^{\infty} a_n (z-z_0)^n = (z-z_0)^N \sum_{n=N}^{\infty} a_n (z-z_0)^{n-N} \\ &= (z-z_0)^N \underbrace{\sum_{n=0}^{\infty} a_{n+N} (z-z_0)^n}_{g(z)} = (z-z_0)^N g(z) \end{aligned}$$

where g is analytic at z_0 and $g(z_0) = a_N \neq 0$. If $N > 0$ we call z_0 a *zero of order* N of f.

It is quite clear that, in general, whenever we can write $f(z) = (z-z_0)^N g(z)$ with g analytic and $g(z_0) \neq 0$ and $N > 0$, that z_0 is a zero of order N. (Multiply the Taylor series of g at z_0 by $(z-z_0)^N$ to see this.)

Pole. If there is some $N > 0$ with $a_{-N} \neq 0$ while $a_n = 0$ for all $n < -N$ then we say z_0 is a *pole of order* N of f.

In this case we can write

$$\begin{aligned} f(z) &= \sum_{n=-N}^{\infty} a_n (z-z_0)^n = (z-z_0)^{-N} \sum_{n=N}^{\infty} a_n (z-z_0)^{n+N} \\ &= (z-z_0)^{-N} \underbrace{\sum_{n=0}^{\infty} a_{n-N} (z-z_0)^n}_{g(z)} = \frac{g(z)}{(z-z_0)^N} \end{aligned}$$

where again, g is analytic at z_0 and $g(z) = a_{-N} \neq 0$.

The situation is similar to that of a zero of order N: If we can write $f(z) = g(z)/(z-z_0)^N$ with g analytic and $g(z_0) \neq 0$ and $N > 0$, that z_0 is a pole of order N.

Essential singularity. If $a_n \neq 0$ for infinitely many $n < 0$, then z_0 is called an *essential singularity* of f.

How to recognize the three kinds of singularity, and a bit about their properties. You don't actually need the Laurent series to recognize the different kinds of singularity.

z_0 is a removable singularity if and only if $|f|$ is bounded in some neighbourhood of z_0.

Proof: The "only if" part is quite obvious: If z_0 is a removable singularity then f is in fact analytic at z_0 (after a suitable redefinition at the single point z_0), and analytic functions, being continuous, are locally bounded.

On the other hand, if $|f|$ is bounded near z_0, define $g(z) = (z-z_0)^2 f(z)$ for $z \neq z_0$ and $g(z_0) = 0$. Then g is analytic at $z \neq z_0$, but also $g'(z_0) = 0$ by direct definition of the derivative. So g is in fact analytic at z_0, and we can write

$$g(z) = \sum_{n=0}^{\infty} b_n (z-z_0)^n$$

for z in a neighbourhood of z_0. Now $b_0 = g(0) = 0$ and $b_1 = g'(0) = 0$, so

$$g(z) = (z-z_0)^2 \sum_{n=2}^{\infty} b_n (z-z_0)^{n-2} = (z-z_0)^2 \underbrace{\sum_{n=0}^{\infty} b_{n+2} (z-z_0)^n}_{f(z)},$$

i.e., the indicated sum must be $f(z)$, which therefore has a removable singularity at z_0. ∎

f has a pole at z_0 if and only if $f(z) \to \infty$ as $z \to z_0$.
Proof: Recall that $f(z) \to \infty$ really means $|f(z)| \to \infty$.

Again, the "only if" part is obvious, for if z_0 is a pole then we can write $f(z) = g(z)/(z-z_0)^N$ where g is analytic with $g(z_0) \neq 0$ and $N > 0$, so $f(z) \to \infty$ follows.

On the other hand, assume that $f(z) \to \infty$ as $z \to z_0$. Now define $h(z) = 1/f(z)$ for $z \neq z_0$. From the assumption it follows that h is bounded in a neighbourhood of z_0 so it has a removable singularity at z_0. Therefore we can write $h(z) = (z-z_0)^N g(z)$ with g analytic, and $g(z_0) \neq 0$, and $N \geq 0$. So $f(z) = 1/\big((z-z_0)^N g(z)\big)$. Since $f(z) \to \infty$ we must in fact have $N > 0$, and so z_0 is a pole. ∎

If course, it follows from the three alternatives and the above two characterizations that z_0 is an essential singularity if and only if $f(z)$ is neither bounded nor goes to infinity as $z \to z_0$. This indicates some rather "wild" behaviour of the function. In fact, more is true:

If z_0 is an essential singularity of f then, for every $\alpha \in \mathbb{C}$ and every neighbourhood of z_0, we can find z in that neighbourhood so that $f(z)$ is arbitrarily close to α.
Proof: We prove the contrapositive. Assume there is some $\alpha \in \mathbb{C}$ and a neighbourhood of z_0 so that $f(z)$ can not get arbitrarily close to α for z in that neighbourhood. But then the function $h(z) = 1/\big(f(z)-\alpha\big)$ is bounded in the given neighbourhood, and therefore it has a removable singularity at z_0. So we can write $1/\big(f(z)-\alpha\big) = (z-z_0)^N g(z)$ with g analytic and $g(z_0) \neq 0$, and $N \geq 0$. Then

$$f(z) = \alpha + \frac{1}{(z-z_0)^N g(z)} = \frac{1+\alpha(z-z_0)^N g(z)}{(z-z_0)^N g(z)}$$

has a removable singularity (if $N = 0$) or else a pole of order N at z_0. ∎

On isolated zeroes. Above we found that if $f(z_0) = 0$, then can write $f(z) = (z-z_0)^N g(z)$ where g is analytic, $g(z_0) \neq 0$ and $N > 0$, or else $f(z)$ is identically zero in some neighbourhood of z_0. In the former case, we call z_0 an *isolated zero* of f.

In other words, if f is analytic at z_0 then either $f(z)$ is identically zero in some neighbourhood of z_0 or else $f(z) \neq 0$ for all z in some neighbourhood of z_0, with the possible exception of $z = z_0$ itself.

Now let f be analytic in a region Ω. (And recall that a region is, by definition, open and connected.) If we write A for the set of points in Ω that are either isolated zeroes, or not zeroes at all, and B for those points where f is identically zero in some neighbourhood, then A and B are both open subsets of Ω. They are also disjoint, and their union is all of Ω. Therefore, since Ω is a region, one of the two sets is empty. It follows that *unless f is identically zero in Ω, then all zeroes of f in Ω are isolated.*