MA2104 Fall 2006, Week 35: Solutions to exercises

Some pictures are at the end.

Problem 1.2.12:

$$\Big|\frac{1+i}{(1-i)(1+3i)}\Big| = \frac{|1+i|}{|1-i|\cdot|1+3i|} = \frac{\sqrt{2}}{\sqrt{2}\cdot\sqrt{10}} = \frac{1}{\sqrt{10}}$$

Problem 1.5.4: Remember that $e^{2ik\pi} = 1$ and $e^{i\pi} = -1$, so $e^{201i\pi} = e^{200i\pi}e^{i\pi} = 1\cdot(-1) = -1$.

Problem 1.5.10: As above. Notice that $701/4 = 175\frac{1}{4} = 174 + \frac{5}{4}$ so that $e^{701i\pi/4} = e^{5i\pi/4} = -e^{i\pi/4} = -\frac{1}{2}\sqrt{2}(1+i) = -\frac{1}{2}\sqrt{2} - \frac{i}{2}\sqrt{2}$.

Problem 1.5.12: We find

$$\Big(-\frac{\sqrt{3}}{2}\Big)^2 + \Big(\frac{1}{2}\Big)^2 = 1$$

so the number has absolute value (modules) one. We're looking for solutions to $\cos\theta = -\frac{1}{2}\sqrt{3}$ and $\sin\theta = \frac{1}{2}$, and $\theta = \pi - \frac{1}{6}\pi = \frac{5}{6}\pi$ fits the bill. So

$$-\frac{\sqrt{3}}{2} + \frac{i}{2} = e^{5i\pi/6}.$$

Problem 1.5.17: Simplify first:

$$\frac{1+i}{1-i} = \frac{1+i}{1-i}\cdot\frac{1+i}{1+i} = \frac{(1+i)^2}{1-i^2} = \frac{1+2i-i^2}{2} = i = e^{i\pi/2}.$$

The reason behind the first trick is that we know that $z\bar{z}$ is real, so we get a real denominator by multiplying and dividing by the conjugate of $1-i$.

Problem 1.5.29: Since $e^{x+iy} = e^x e^{iy}$ has absolute value $|e^{x+iy}| = e^x$, and by assumption $x_1 \le x \le x_2$, we find $e^{x_1} \le |e^{x+iy}| \le e^{x_2}$ so that e^{x+iy} lies between the circles of radius e^{x_1} and e^{x_2} respectively.

Similarly, $Arg e^{x+iy} = y$ so long as $-\pi < y \le \pi$. In this case, $0 \le \alpha_1 \le y \le \alpha_2 \le \pi$, so certainly the argument lies between α_1 and α_2 as in the picture.

I think the argument that the image fills out the entire sector shown is easy enough?

Problem 1.6.22: If you pick purely imaginary values for z, that is $z = iy$ with $y \in \mathbb{R}$, then $\cos z = \cos iy = \cosh y$ which is unbounded. Similarly $\sin iy = i\sinh y$ which is also unbounded.

Problem 1.6.40:

$$\cosh^2 z - \sinh^2 z = \Big(\frac{e^z + e^{-z}}{2}\Big)^2 - \Big(\frac{e^z - e^{-z}}{2}\Big)^2 = \frac{e^{2z} + 2 + e^{-2z}}{4} - \frac{e^{2z} - 2 + e^{-2z}}{4} = 1.$$

Problem 1.7.2: Write the number on polar form. $|-3-3i| = 3\sqrt{2}$, and $(-3-3i)/|-3-3i| = -\frac{1}{2}\sqrt{2}(1+i) = e^{5i\pi/4}$. So $\ln(-3-3i) = \ln(3\sqrt{2}) + i\arg(-3-3i) = \ln(3\sqrt{2}) + 5i\pi/4 + 2ki\pi$, for some (any) $k \in \mathbb{Z}$.

Problem 1.7.14: Write $1 + i = \sqrt{2}e^{i\pi/4}$, so taking logarithms we get $-z = \ln\sqrt{2} + i\pi/4 + 2ki\pi$, that is $z = -\frac{1}{2}\ln 2 + i\pi/4 - 2ki\pi$ for $k \in \mathbb{Z}$.

Problem 1.7.19b: By definition $\operatorname{Ln} w = \ln|w| + i\operatorname{Arg} w$, and so $-\pi < \operatorname{Im}\operatorname{Ln} w \le \pi$ for all w. In particular, $\operatorname{Ln} e^z = z$ implies $-\pi < \operatorname{Im} z \le \pi$.

Conversely, note that $e^z = e^{\operatorname{Re} z}e^{i\operatorname{Im} z}$ has argument $\operatorname{Im} z$. If $-\pi < \operatorname{Im} z \le \pi$ then this must be the principal value of the argument, i.e., $\operatorname{Arg} e^z = \operatorname{Im} z$. Thus $\operatorname{Ln} e^z = \ln|e^z| + i\operatorname{Im} z = \ln e^{\operatorname{Re} z} + i\operatorname{Im} z = \operatorname{Re} z + i\operatorname{Im} z = z$.

Problem 1.7.31: The given equation has these equivalent forms:

$$\begin{aligned}
\cos z &= \sin z \\
\frac{e^{iz}+e^{-iz}}{2} &= \frac{e^{iz}-e^{-iz}}{2i} \\
i(e^{iz}+e^{-iz}) &= e^{iz}-e^{-iz} \\
i(e^{2iz}+1) &= e^{2iz}-1 \\
1+i &= (1-i)e^{2iz} \\
e^{2iz} &= \frac{1+i}{1-i} = i = e^{i\pi/2} \qquad \text{see problem 1.5.17} \\
2iz &= \frac{i}{2}\pi + 2k, \qquad k \in \mathbb{Z} \\
z &= \frac{1}{4}\pi - ik, \qquad k \in \mathbb{Z}.
\end{aligned}$$

Problem 1.7.32a: We define $w = \arccos z$ by solving $\cos w = z$ for w. Rewrite as follows:

$$\begin{aligned}
\cos w &= z \\
e^{iw} + e^{-iw} &= 2z \\
e^{2iw} - 2ze^{iw} &= -1 \\
\bigl(e^{iw} - z\bigr)^2 &= -1 + z^2 \\
e^{iw} &= z \pm \sqrt{z^2-1} \\
iw &= \ln\bigl(z \pm \sqrt{z^2-1}\bigr) \\
w &= -i\ln\bigl(z \pm \sqrt{z^2-1}\bigr)
\end{aligned}$$

The book does not write the $\pm$ in front of the square roots, but then it should be implicit that either branch of the square root may be chosen. And after that, there are infinitely many branches of the logarithm to choose from.

Good choices of the branches for the arccos function are not at all obvious, except for $z = x$ real and between -1 and 1. Then the number in the square root is negative, so it is better to write $w = -i\ln\bigl(x \pm i\sqrt{1-x^2}\bigr)$. The expression in the logarithm lies on the upper half of the unit circle, so if we chose the principal branch of the logarithm, w works out to lie in $[0,\pi]$, just like the arccos of a real number in $[-1,1]$ should do.

Problem 2.1.4: $\bigl\{z\colon 1 < |z-i| \le 2\bigr\}$ has interior $\bigl\{z\colon 1 < |z-i| < 2\bigr\}$ and boundary $\bigl\{z\colon |z-i| = 1 \text{ or } |z-i| = 2\bigr\}$.

Problem 2.1.8: $\bigl\{z\colon z \ne 0,\ |\operatorname{Arg} z| < \frac{1}{4}\pi\bigr\}$ is open, connected (and hence a region), but not closed.

Problem 2.1.17: A boundary point of S is defined by the requirement that every neighbourhood contains some points of S and some points from the complement. Since this requirement is symmetric in S and its complement (it does not change when S is replaced by its complement), then S and $\mathbb{C} \setminus S$ have the same boundary.

If S is open then any point in S has a neighbourhood contained in S, and so that point is not a boundary point. Conversely, if S is not open then some point $z \in S$ is such that every neighbourhood of z contains a point not in S. Since z itself belongs to any neighbourhood, then z is a boundary point.

In other words, S is open if and only if it contains no boundary point. In other words, it is open if and only if the boundary is contained in the complement, which means the complement is closed.

(There must be lots of different ways to organize the above reasoning. Don't assume that yours is wrong just because it didn't look like mine.)

Problem 2.1.21: First, if A and B are open then $A \cup B$ is open. For if $z \in A \cup B$ then $z \in A$ or $z \in B$. If $z \in A$ then, since A is open, some neighbourhood of z is contained in A. But then that neighbourhood is also contained in $A \cup B$. The same applies if $z \in B$.

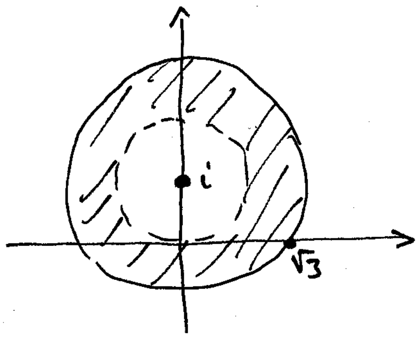
Second, if A and B are connected and $z_0 \in A \cap B$ then there is a path in either A or in B to any given point in $A \cup B$, since both sets are connected. Thus there is a path in $A \cup B$ from z_0 to any other point in $A \cup B$. So we can find a path in $A \cup B$ between any two points, by going from one of them to z_0 and from there to the other one.

Problem 2.2.6: $\operatorname{Arg} z$ is bounded, since $|\operatorname{Arg} z| \leq \pi$. The squeeze law thus guarantees that $z \operatorname{Arg} z \to 0$ when $z \to 0$.

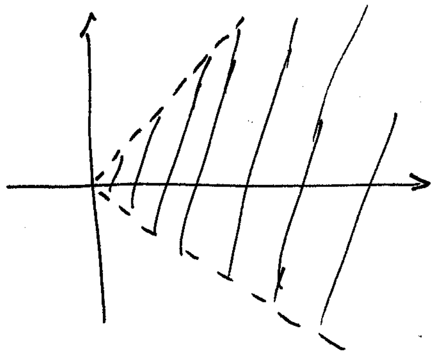
Problem 2.2.9: Roughly speaking, $\operatorname{Arg} z \to \pi$ if z approaches the negative real axis from above, while $\operatorname{Arg} z \to -\pi$ if it does so from below. In either case, $(\operatorname{Arg} z)^2 \to \pi^2$, so we should have $\lim_{z \to -3} (\operatorname{Arg} z)^2 = \pi^2$.

Problem 2.2.23: When $z \to 0$ along the real axis, then $1/z \to \pm\infty$, with the plus sign if we come from the right, and the minus sign if we come from the left. In the former case $e^{1/z} \to \infty$, and in the latter case, $e^{1/z} \to 0$. So the given limit does not exist. (Also, with an imaginary valu $z = iy$, $e^{1/z} = e^{-i/y}$ which has absolute value 1, but spins round the unit circle infinitely often as $y \to 0$ from either side.)

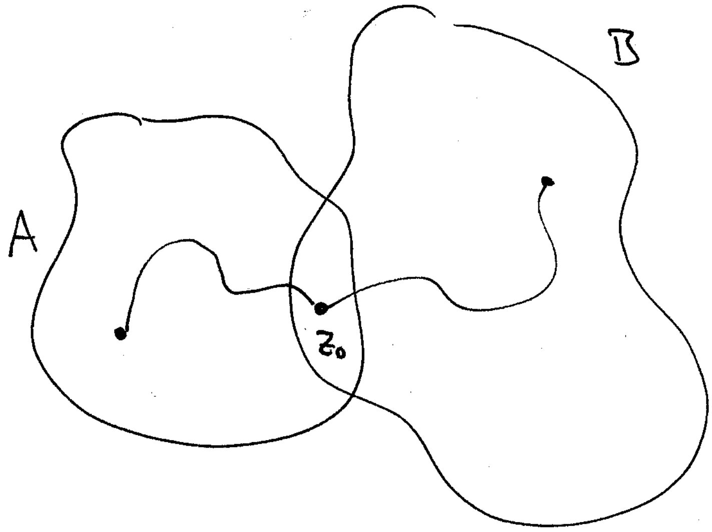
2.1.4



2.1.8



2.1.21



If A and B are open, I ought to draw the boundaries dotted, but I am too lazy.

2.2.9