MA2104 Fall 2006, Week 36: Solutions to exercises

Problem 2.3.5: We can differentiate this using standard rules:

$$\frac{d}{dz}\frac{1}{z^3+1} = \frac{-3z^2}{(z^3+1)^2}, \qquad z \notin \left\{-1, \tfrac{1}{2}\sqrt{3}+\tfrac{i}{2}, \tfrac{1}{2}\sqrt{3}-\tfrac{i}{2}\right\}.$$

The stated exceptions are the points where $z^3 = -1$, which happens at $z = e^{i(\pi/3+2j\pi/3)}$, where $j = 0, 1, 2$. The case $j = 1$ gives $z = e^{\pi} = -1$, while $j = 0$ gives $z = \cos\frac{\pi}{3} + i\sin\frac{\pi}{3} = \frac{1}{2}\sqrt{2} + \frac{i}{2}$. The case $j = 2$ produces $z = \frac{1}{2}\sqrt{2} - \frac{i}{2}$ similarly.

Problem 2.3.9: Write $z^{2/3} = e^{(2/3)\ln z}$ and differentiate:

$$\frac{d}{dz}z^{2/3} = e^{(2/3)\ln z}\cdot\frac{2}{3z} = \tfrac{2}{3}e^{(2/3)\ln z}e^{-\ln z} = \tfrac{2}{3}e^{(2/3)\ln z-\ln z} = \tfrac{2}{3}e^{-(1/3)\ln z} = \tfrac{2}{3}z^{-1/3}$$

which is correct so long as one uses the *same branch of the logarithm* the whole way (in particular, using the principal branch is fine).

This simple principle works for all powers of the form z^α, with the same proof. Another proof is given in problem 2.3.21.

Problem 2.3.15: Rewrite:

$$\lim_{z\to 0}\left(\frac{1}{z\sqrt{1+z}} - \frac{1}{z}\right) = \lim_{z\to 0}\frac{\dfrac{1}{\sqrt{1+z}} - 1}{z} = \lim_{z\to 0}\frac{\dfrac{1}{\sqrt{1+z}} - \dfrac{1}{\sqrt{1+0}}}{z}$$
$$= \frac{d}{dz}\frac{1}{\sqrt{1+z}}\bigg|_{z=0} = -\tfrac{1}{2}(1+z)^{-3/2}\bigg|_{z=0} = -\tfrac{1}{2}$$

where we have used the choice of the principal branch at the end. (See also the remark at the end of the previous solution.)

Problem 2.3.21: We use the identity

$$\left(z^{p/q}\right)^q = z^p. \tag{1}$$

Since p and q are integers, only the inner $z^{p/q}$ suffers from multiple values, and the identity is easily proven

$$\left(z^{p/q}\right)^q = \left(e^{(p/q)\ln z}\right)^q = e^{p\ln z}$$

Here we relied on the identidy $(e^w)^q = e^{qw}$, which is proved using induction on q.

To use Theorem 2.3.4 (Asmar p. 96) with this identity, put $g(z) = z^{p/q}$, $f(w) = w^q$, and $h(z) = z^p$. The above identity states $h(z) = f(g(z))$.

So long as we are working in a region where $g(z) = z^{p/q}$ has a continuous branch and $z \neq 0$, the conditions of Theorem 4 are satisfied (in particular $f'\big(g(z)\big) = q\big(g(z)\big)^{q-1} \neq 0$), so the conclusion of the theorem is that g is differentiable, and

$$\frac{d}{dz}z^{p/q} = g'(z) = \frac{h'(z)}{f'\big(g(z)\big)} = \frac{pz^{p-1}}{q\big(z^{p/q}\big)^{q-1}}.$$

Here we must pause to point out that the general rules $w^{m+n} = w^m w^n$ and $\big(w^m\big)^n = w^{mn}$ is never problematic for integers m and n, but they easily produce wrong results when used with non-integers. However, the special case (1) still holds, so we find

$$\frac{d}{dz}z^{p/q} = \frac{pz^{p-1}}{q\big(z^{p/q}\big)^q\big(z^{p/q}\big)^{-1}} = \frac{p}{q}\frac{z^p z^{-1}}{z^p\big(z^{p/q}\big)^{-1}} = \frac{p}{q}\frac{z^{p/q}}{z}.$$

As above, this can be further simplified to $\alpha z^{\alpha-1}$ where $\alpha = p/q$, provided the powers z^α and $z^{\alpha-1}$ are computed using the same branch of the logarithm.

Problem 2.3.27: Since $g(z_0) = 0$ but $g'(z_0) = 0$, there is a punctured neighbourhood of z_0 in which $g(z) \neq 0$: For

$$\frac{g(z)}{z - z_0} = \frac{g(z) - g(z_0)}{z - z_0} \to g'(z_0) \neq 0$$

(convergence as $z \to z_0$) implies that the lefthand side must be nonzero when $|z - z_0|$ is small enough.

Thus we never risk divison by zero in the following calculation, so long as $|z - z_0|$ is small enough (and nonzero of course):

$$\frac{f(z)}{g(z)} = \frac{\dfrac{f(z) - f(z_0)}{z - z_0}}{\dfrac{g(z) - g(z_0)}{z - z_0}} \to \frac{f'(z_0)}{g'(z_0)}, \qquad \text{when } z \to z_0.$$

Problem 2.3.28: (a) Since $(i^2 + 1)^7 = 0$ and $i^6 + 1 = 0$, we can use L'Hospital:

$$\lim_{z \to i} \frac{(z^2 + 1)^7}{z^6 + 1} = \frac{14z(z^2 + 1)^6}{6z^5}\bigg|_{z=i} = \frac{14i(i^2 + 1)^6}{6i^5} = 0$$

(b) Verify that $i^3 + (1 - 3i)i^2 + (i - 3)i + 2 + i = 0$ (and $i - i = 0$ of course) and use L'Hospital:

$$\begin{aligned}\lim_{z \to i} \frac{z^3 + (1 - 3i)z^2 + (i - 3)z + 2 + i}{z - i} &= \frac{3z^2 + 2(1 - 3i)z + i - 3}{1}\bigg|_{z=i} \\ &= -3 + 2(1 - 3i)i + i - 3 = 3i.\end{aligned}$$

Problem 2.4.6: We find

$$\begin{aligned} u &= \frac{y}{x^2 + y^2}, & v &= \frac{-x}{x^2 + y^2}, \\ u_x &= \frac{-2xy}{(x^2 + y^2)^2}, & v_y &= \frac{2xy}{x^2 + y^2}, \\ u_y &= \frac{x^2 - y^2}{(x^2 + y^2)^2}, & v_x &= \frac{y^2 - x^2}{x^2 + y^2}. \end{aligned}$$

Thus we have $u_x = -v_y$ and $u_y = v_x$, which *looks* like the Cauchy–Riemann equation, except the minus sign is in the wrong equation. Therefore the real Cauchy–Riemann equations $u_x = v_y$ and $u_y = -v_x$ are satisfied only where $xy = 0$ and $x^2 = y^2$, respectively, and both are satisfied only where $x = y = 0$. Oops, not even there, for then we divide by zero.

Thus the given function is nowhere differentiable.

This may come as no surprise if we rewrite a bit: When $z = x + iy$ then $-ix + y = -i(x + iy) = -iz$, so the function under consideration is

$$\frac{-iz}{|z|^2} = \frac{-iz}{z\bar{z}} = \frac{-i}{\bar{z}}$$

and we already know that $z \mapsto \bar{z}$ is not analytic.

Problem 2.4.31: We can do better than the book, and define any branch of the inverse tangent by using any branch of the logarithm:

$$\arctan z = \frac{i}{2} \ln \frac{1 - iz}{1 + iz}$$

which we differentiate using the chain rule:

$$\frac{d}{dz} \arctan z = \frac{i}{2} \frac{1 + iz}{1 - iz} \cdot \frac{-i(1 + iz) - i(1 - iz)}{(1 + iz)^2} = \frac{1}{(1 - iz)(1 + iz)} = \frac{1}{1 + z^2}$$

just as it should be.

Problem 2.4.33: (a) The book gives two formulas for f' – equations (3) and (5) on p. 101:

$$f' = u_x + iv_x = v_y - iu_y.$$

Given the second Cauchy–Riemann equation $u_y = -v_x$, it is of course trivial to rewrite these as

$$f' = u_x - iu_y = v_y + iv_x.$$

(b) The stated identity follows at once from the above and the identity $|a+ib|^2 = a^2 + b^2$ when a and b are real.

(c) If u or v is constant in Ω, then $f' = 0$ follows. Thus f is constant. (This requires the connectedness of Ω, which is part of the definition of Ω being a region.)

Problem 2.4.35: With $f_x = u_x + iv_x$ and $f_y = u_y + iv_y$ we find

$$f_x + if_y = u_x + iv_x + i(u_y + iv_y) = u_x - v_y + i(v_x + u_y),$$

which is zero precisely when $u_x - v_y = 0$ and $v_x + u_y = 0$. These are the Cauchy–Riemann equations, rearranged.

Problem 2.4.38: (a) Since $|f|^2 = u^2 + v^2$, it is clear that $|f|$ is constant if and only if $u^2 + v^2$ is constant. If $u^2 + v^2 = 0$ then $f = 0$, so there is nothing to prove.

(b) Differentiating $u^2 + v^2 = c$ first wrt x and then wrt y (and dividing both equations by 2), we get

$$uu_x + vv_x = 0, \quad uu_y + vv_y = 0.$$

Substitute $v_x = -u_y$ and $v_y = u_x$ from the Cauchy–Riemann equations to get

$$uu_x - vu_y = 0, \quad uu_y + vu_x = 0.$$

(c) These equations can be written

$$\begin{pmatrix} u & -v \\ v & u \end{pmatrix} \begin{pmatrix} u_x \\ u_y \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}.$$

The determinant of the matrix on the left is $u^2 + v^2 = c > 0$, so we must have $u_x = u_y = 0$.

(d) We can simply refer to problem 2.4.33 to conclude that f is constant.