## MA2104 Fall 2006, Week 39: Solutions to exercises

**Problem 3.6.1:** Cauchy's formula for the cosine function applied to the closed path $C_1(0)$ and the point $z = 0$, which is inside it:

$$\cos 0 = \frac{1}{2\pi i} \int_{C_1(0)} \frac{\cos \zeta}{\zeta - 0} \, d\zeta.$$

This is OK because the cosine function is entire, so in particular it is analytic on and inside $C_1(0)$. Multiply by $2\pi i$ and change the name of the integration variable to get

$$\int_{C_1(0)} \frac{\cos z}{z - 0} \, dz = 2\pi i.$$

(This was much too detailed; in the next problems, I shall assume that the name of the integration variable causes no difficulty.)

**Problem 3.6.2:** This uses the Cauchy formula for the point $i$, which is inside $C_3(0)$, and applied to the function $f(z) = e^{z^2} \cos z$:

$$\int_{C_3(0)} \frac{e^{z^2} \cos z}{z - i} \, dz = 2\pi i e^{i^2} \cos i = 2\pi i e^{-1} \frac{e^{i^2} + e^{-i^2}}{2} = 2\pi i e^{-2}.$$

**Problem 3.6.3:** Factor the denominator in the integrand. Find its zeros by the formula for the solution of the quadratic equation, or just complete the square: $z^2 - 5z + 4 = (z - \frac{5}{2})^2 - \frac{9}{4} = (z - \frac{5}{2} + \frac{3}{2})(z - \frac{5}{2} - \frac{3}{2}) = (z-1)(z-4)$. Of the two zeros $z = 1$ and $z = 4$, the former lies inside the circle $C_2(1)$, and the latter outside it: So the integral can be viewed as the Cauchy integral formula applied to the function $f(z) = 1/(z-4)$ and the point $z = 1$:

$$\begin{aligned} \frac{1}{2\pi i} \int_{C_2(1)} \frac{1}{z^2 - 5z + 4} \, dz &= \frac{1}{2\pi i} \int_{C_2(1)} \frac{1}{(z-1)(z-4)} \, dz \\ &= \frac{1}{2\pi i} \int_{C_2(1)} \frac{1/(z-4)}{z-1} \, dz = \frac{1}{1-4} = -\frac{1}{3}. \end{aligned}$$

**Problem 3.6.17:** We would like to factor the denominator again. This time it is a cubic polynomial, with the factorization

$$z^3 - 3z + 2 = (z-1)^2(z+2).$$

We could have found that out just by looking for a rational root: If the polynomial has one, the root must be an integer (the denominator divides the coefficient of the highest order term), and that integer must divide 2. So the numbers $\pm 1$ and $\pm 2$ are the only possible candidates for a rational root. We try them all, and find that the rational roots are $z = 1$ and $z = -2$. Either polynomial division or the realization that $z = 1$ must be a double root because it is also a root of the derivative $3z^2 - 3$ finishes the factorization effort.

Next we find out the location of the roots $z = 1$ and $z = -2$ relative to the integration path $C_{3/2}(0)$: Clearly, $z = 1$ is inside and $z = -2$ is outside. So this will look like Cauchy's integral formula applied for the *derivative* of the function $f(z) = 1/(z+2)$:

$$\begin{aligned} \int_{C_{3/2}(0)} \frac{1}{z^3 - 3z + 2} \, dz &= \int_{C_{3/2}(0)} \frac{1}{(z-1)^2(z+2)} \, dz = \int_{C_{3/2}(0)} \frac{f(z)}{(z-1)^2} \, dz \\ &= \frac{2\pi i}{1!} f'(1) = -\frac{2\pi i}{(z+2)^2}\bigg|_{z=1} = -\frac{2\pi i}{9}. \end{aligned}$$

**Problem 3.6.20:** Here we get the factorization $(z^4 - 1) = (z-1)(z+1)(z-i)(z+i)$, but since all the zeros lie inside the integration path $C_2(0)$ we cannot use the trick of the previous two questions.

However we can use partial fraction decomposition instead. (*Note: This method could have been used on problems 3.6.3 and 3.6.17 too.*)

In fact, remembering back to problem 3.4.33, we see immediately that the answer must be zero!

But we can verify that by computing the coefficients of the partial fraction decomposition

$$\frac{1}{z^4-1} = \frac{1}{(z-1)(z+1)(z-i)(z+i)} = \frac{A}{z-1} + \frac{B}{z+1} + \frac{C}{z-i} + \frac{D}{z+i}.$$

We can save a bit of work by doing it in two steps: Put $z^2 = w$ and note that

$$\begin{aligned}\frac{1}{z^4-1} &= \frac{1}{w^2-1} = \frac{1}{2}\Big(\frac{1}{w-1} - \frac{1}{w+1}\Big) = \frac{1}{2}\Big(\frac{1}{z^2-1} - \frac{1}{z^2+1}\Big) \\ &= \frac{1}{4}\Big(\frac{1}{z-1} - \frac{1}{z+1} + \frac{i}{z-i} - \frac{i}{z+i}\Big)\end{aligned}$$

so that

$$\begin{aligned}\int_{C_2(0)} \frac{1}{z^4-1}\,dz &= \frac{1}{4}\int_{C_2(0)} \Big(\frac{1}{z-1} - \frac{1}{z+1} + \frac{i}{z-i} - \frac{i}{z+i}\Big)\,dz \\ &= \frac{2\pi i}{4}(1 - 1 + i - i) = 0.\end{aligned}$$

**Problem 3.6.21:** (a) The integrand is an analytic function of $z$ for $z$ in the unit disk, and it is continuous as a function of $z$ and $t$. So the integral is also analytic.

(b) With $\gamma(t) = \zeta = e^{it}$ we find $d\zeta = ie^{it}\,dt$, and we recognize the given integral as a path integral:

$$\frac{1}{2\pi}\int_0^{2\pi} \frac{e^{it}}{e^{it}-z}\,dt = \frac{1}{2\pi i}\int_0^{2\pi} \frac{ie^{it}}{e^{it}-z}\,dt = \frac{1}{2\pi i}\int_\gamma \frac{1}{\zeta - z}\,d\zeta = 1$$

**Problem 3.6.28:** (a) It's just Cauchy's integral formula for the exponential function:

$$\frac{1}{2\pi i}\int_{C_1(0)} \frac{e^z}{z}\,dz = e^0 = 1.$$

(b) With the parametrization as in problem 3.6.21, $\gamma(t) = z = e^{it}$, we write that as

$$\frac{1}{2\pi}\int_0^{2\pi} \frac{e^{e^{it}}e^{it}}{e^{it}}\,dt = 1, \quad \text{which simplifies into} \quad \int_0^{2\pi} e^{e^{it}}\,dt = 2\pi.$$

Next,

$$e^{e^{it}} = e^{\cos t + i\sin t} = e^{\cos t}(\cos\sin t + i\sin\sin t),$$

which we will substitute in the above formula. The functions we are integrating are periodic with period $2\pi$. So we may replace the limits of the integral as well:

$$\int_{-\pi}^{\pi} e^{\cos t}(\cos\sin t + i\sin\sin t)\,dt = 2\pi.$$

The term $e^{\cos t}\sin\sin t$ is an odd function of $t$, so its integral will be zero. The term $e^{\cos t}\cos\sin t$ is odd, so its integral is twice the integral from 0 to $\pi$, and we have arrived at the desired conclusion

$$\int_0^{\pi} e^{\cos t}\cos\sin t\,dt = \pi.$$

**Problem 3.6.31:** Do a partial fraction decomposition:

$$\frac{1}{(\zeta - z)\zeta} = \frac{1}{z}\Big(\frac{1}{\zeta - z} - \frac{1}{\zeta}\Big),$$

substitute this in the integral, and use Cauchy's integral formula twice.

**Problem 3.7.3:** $f(z) = e^{-z^2}$ is analytic, so the maximum and minimum values of $|f(z)|$ must happen on the boundary. (The only exception is if $f(z) = 0$ somewhere, but this doesn't happen.) The boundary here is the union of the two circles $|z| = 1$ and $|z| = 2$. We also note that $|e^{-z^2}| = e^{-\operatorname{Re} z^2} = e^{y^2 - x^2}$, which is maximal on a circle where $x = 0$, and minimal where $y = 0$. (In fact, if $|z| = x^2 + y^2 = r^2$ then $e^{r^2 - 2x^2}$.) The most extreme values clearly happen on the circle $z = 2$, with the maximum value $e^4$ at $z = \pm 2i$ and the minimum value $e^{-4}$ at $z = \pm 2$.

We should not be surprised that the extreme values happen on the *outer* circle, since $f$ is not analytic not only in the annulus, but in the whole disk $|z| \le 2$.

**Problem 3.7.9:** We find $|\operatorname{Ln} z| = |\ln|z| + i \operatorname{Arg} z| = \sqrt{(\ln|z|)^2 + (\operatorname{Arg} z)^2}$, so it doesn't take a lot of theory to realize that the maximum happens where both $|z|$ and $|\operatorname{Arg} z|$ are maximal, while the minimum happens where these are both minimal. So the maximum value $\sqrt{(\ln 2)^2 + (\frac{\pi}{4})^2}$ happens at $z = 2e^{i\pi/4} = \sqrt{2}(1 + i)$, and the minimum value 0 happens at $z = 1$.

**Problem 3.7.11:** Note that $|e^{e^z}| = e^{\operatorname{Re} e^z} = e^{e^{\operatorname{Re} z} \cos \operatorname{Im} z} = e^0 = 1$ when $\operatorname{Im} z = \pm\frac{\pi}{2}$. But in the middle of the strip, $z$ is real, so $e^z$ is real and goes to infinity as $z \to +\infty$, and then $e^{e^z} \to +\infty$ as well. So this function is very unbounded in the strip, even though it is bounded on its boundary. This does not contradict the maximum modulus principle because the region is unbounded.