MA2104 Fall 2006, Week 40: Solutions to exercises

Problem 3.7.14: From $\lim_{z\to\infty}|f(z)| = c$ we find (using $\epsilon = 1$ in the definition of limit) some M so that $\big||f(z)|-c\big| < 1$ for $|z| > M$. Clearly then, for such z we get $|f(z)| < c+1$ (which we interpret as "$f(z)$ is bounded near infinity").

On the other hand $\{z\colon |z| \le M\}$ is compact (closed and bounded) so the continuous function f is bounded on this set, say $|f(z)| \le C$ when $|z| \le M$.

Then in all cases, $|f(z)| \le \max(C, c+1)$.

Problem 3.7.15: Assume f is entire and $f(z) \to 0$ as $z \to \infty$. It follows from problem 3.7.14 that f is bounded. So, by Liouville's theorem, it is constant: Say, $f(z) = C$ for all $z \in \mathbb{C}$. Letting $z \to \infty$, we conclude that $C = 0$.

Problem 3.7.19: We are assuming f is entire and $f(z)/z \to 0$ as $z \to \infty$. We are asked to show that f is constant.

This seems stronger than the Liouville theorem, since the assumption is weaker than assumption in Liouville's theorem. We could imagine, for example, that $|f(z)|$ is approximately equal to $|z|^{1/2}$ for large $|z|$. But the result we are to prove shows that no entire function like that can exist. The result is also quite *sharp*, as the example $f(z) = z$ shows.

First proof: Use Cauchy's generalized formula

$$f'(z) = \frac{1}{2\pi i}\int_\gamma \frac{f(\zeta)}{(\zeta - z)^2}\,d\zeta$$

where γ is a closed path around z. Say $\gamma = C_R(0)$. Given $\epsilon > 0$ we can find M so that $|f(z)/z| < \epsilon$ whenever $|z| > M$.

If $R > \max(M, |z|)$ then for ζ on $C_R(0)$, $|f(\zeta)| < \epsilon|\zeta| = \epsilon R$. Also $|\zeta - z|^2 > (R - |z|)^2$. The length of the integration path $C_R(0)$ is $2\pi R$. Altgother, then, estimating the above integral we find

$$|f'(z)| < \frac{2\pi R}{|2\pi i|}\frac{\epsilon R}{(R-|z|)^2} = \frac{\epsilon R^2}{(R-|z|)^2} < 4\epsilon$$

where we choose $R > 2|z|$ for the final inequality. Since $\epsilon > 0$ was arbitrary, $f'(z) = 0$. And since this holds for all z, f is constant.

Second proof: Recall (Theorem 4, p. 206) that the function $g(z) = \big(f(z) - f(0)\big)/z$ is analytic around $z = 0$, if you just define $g(0) = f'(0)$. But g is clearly analytic everywhere else too, so it is entire. It follows from the assumption that $g(z) = f(z)/z - f(0)/z \to 0$ as $z \to \infty$. So problem 3.7.15 shows that $g(z) = 0$ for all z. Thus f is constant.

Problem 3.7.20: Just as in the second proof for problem 3.7.19, $g(z) = \big(f(z) - f(0)\big)/z$ is entire and bounded, and therefore constant. Since in fact $g(z) \to c$ when $z \to \infty$, that constant is c. Solving the equation $\big(f(z) - f(0)\big)/z = c$ for $f(z)$ we get the desired formula, with $b = f(0)$.

Problem 4.2.1:

$$\lim_{n\to\infty} f_n(x) = \lim_{n\to\infty}\frac{\sin nx}{n} = 0, \qquad 0 < x < \pi$$

because $|(\sin nx)/n| \le 1/n \to 0$. This also shows the convergence is uniform, because $1/n$ is independent of x.

To be overly pedantic, given $\epsilon > 0$ pick N so that $1/n < \epsilon$ whenever $n \ge N$. Then the above inequality shows that $|f_n(x)| < \epsilon$ for all x, when $n \ge N$. This is just what uniform convergence to zero means.

Problem 4.2.2:
$$\lim_{n\to\infty} f_n(x) = \lim_{n\to\infty} \frac{\sin nx}{nx} = 0, \qquad 0 < x < \pi$$
because $|(\sin nx)/n| \le 1/(nx) \to 0$ for any given x.

But since the estimate $1/(nx)$ is not independent of x, we cannot conclude that the convergence is uniform. Neither can we (yet) conclude it is *not* uniform, for there is a possibility that a better estimate could give us what we want.

However, the convergence *is* not uniform. One way to see this is to note that
$$\lim_{x\to 0} f_n(x) = \lim_{x\to 0} \frac{\sin nx}{nx} = 1, \qquad n = 1, 2, 3, \dots$$
This means, in particular, that for any n there is some x (near 0) so that $f_n(x) \ge \frac{1}{2}$.

But if $f_n(x) \to 0$ uniformly, there should be some N so that (and here we pick $\epsilon = \frac{1}{2}$ in the definition of uniform convergence) $f_n(x) < \frac{1}{2}$ for every x and every $n \ge N$. This is clearly contradicted by the previous paragraph.

Another way is to just look for places where $f_n(x)$ is large. Given any n we can pick x by setting $nx = \pi/2$. Then $f_n(x) = 2/\pi$. Again this contradicts the definition of uniform convergence to zero, this time with $\epsilon = 2/\pi$.

We are asked to find a suitable interval where the sequence does converge uniformly. Since the problem arose near $x = 0$, it seems reasonable to omit small values of x. At the start, we found the estimate $|f_n(x)| \le 1/(nx)$. If we pick any $\delta > 0$ and insists on only considering $x \ge \delta$, the estimate implies $|f_n(x)| \le 1/(n\delta)$ which *is* independent of x, and clearly $1/(n\delta) \to 0$ as $n \to \infty$, so we have uniform convergence to 0 on $[\delta, \infty)$ for any $\delta > 0$.

Problem 4.2.13: We find
$$\left|\frac{z^n}{n(n+1)}\right| < \frac{1}{n(n+1)} = \frac{1}{n} - \frac{1}{n+1}, \qquad |z| \le 1.$$
and
$$\sum_{n=1}^{\infty} \left(\frac{1}{n} - \frac{1}{n+1}\right) = 1 < \infty$$
(this is a *telescoping* series), so the Weierstrass M-test shows that the given series converges uniformly on the given set.

As an alternative, one can use
$$\frac{1}{n(n+1)} < \frac{1}{n^2}$$
and the fact that $\sum n^{-2}$ is finite, as follows from the integral test. In fact, the sum is $\pi^2/6$. We shall show this later using a Fourier series.

Problem 4.2.18: In order to find a good upper bound on $|1/(5-z)^n|$ we need a lower bound on $|(5-z)^n|$, and therefore on $|5-z|$. We are given $|z| \le \frac{7}{2}$, so we use $|5-z| \ge 5 - |z| \ge 5 - \frac{7}{2} = \frac{3}{2}$. Therefore $|(5-z)^n| \ge \left(\frac{3}{2}\right)^n$, so $|(5-z)^{-n}| \le \left(\frac{2}{3}\right)^n$, and we are done since $\sum\left(\frac{2}{3}\right)^n = 1/(1-\frac{2}{3}) = 3 < \infty$.

Problem 4.2.25: (a) Yes, since $|z - \frac{1}{2}| < \frac{1}{6}$ implies $|z| = |z - \frac{1}{2} + \frac{1}{2}| \le |z - \frac{1}{2}| + |\frac{1}{2}| < \frac{1}{6} + \frac{1}{2} = \frac{2}{3}$ so that $|z^n| < \left(\frac{2}{3}\right)^n$ and $\sum\left(\frac{2}{3}\right)^n = 1/(1-\frac{2}{3}) = 3 < \infty$.

(b) If we try to repeat the success from (a) the best estimate we get is $|z| < \frac{1}{2} + \frac{1}{2} = 1$, and the proof breaks down. In fact, given only the requirement $|z - \frac{1}{2}| < \frac{1}{2}$ then we can get z as close to 1 as we wish, and this seems to get in the way of uniform convergence.

In this case we are fortunate that we can compute things explicitly: For a tail of the series, we find
$$\sum_{n=N}^{\infty} z^n = \frac{z^{N+1}}{1-z} \to \infty \qquad \text{as } z \to 1,$$

but if we had uniform convergence, the tails of the sequence should become uniformly small for all z, when N becomes large. So the series is not uniformly convergent on the region $|z - \frac{1}{2}| < \frac{1}{2}$.

Problem 4.4.21: We find

$$\left|\frac{(z-2)^n}{3^n}\right| \leq \left|\frac{2.9^n}{3^n}\right| \quad \text{and} \quad \left|\frac{2^n}{(z-2)^n}\right| \leq \left|\frac{2^n}{2.01^n}\right|,$$

so

$$\left|\frac{(z-2)^n}{3^n} + \frac{2^n}{(z-2)^n}\right| \leq \left(\frac{2.9}{3}\right)^n + \left(\frac{2}{2.01}\right)^n,$$

and

$$\sum_{n=0}^{\infty}\left(\left(\frac{2.9}{3}\right)^n + \left(\frac{2}{2.01}\right)^n\right) = 231 < \infty$$

(except who cares about the exact sum anyway), and we're done.