Stability for linear systems

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The purpose of this text is to correct an error in Perko's book (see the remarks and reference at the end).

Theorem 1 *Given a real $n \times n$ matrix A, the following are equivalent:*

(a) $\lim_{t\to\infty} e^{At}x_0 = 0$ *for every* $x_0 \in \mathbb{R}^n$
(b) $\lim_{t\to\infty} \|e^{At}\| = 0$
(c) *There exist constants* $a > 0$ *and* M *so that* $|e^{At}x_0| \le Me^{-at}|x_0|$ *for all* $t \ge 0$ *and* $x_0 \in \mathbb{R}^n$
(d) *Every eigenvalue of* A *har real part* < 0
(e) $\lim_{t\to-\infty} |e^{At}x_0| = \infty$ *for every nonzero* $x_0 \in \mathbb{R}^n$
(f) *There exist constants* $a > 0$ *and* $m > 0$ *so that* $|e^{At}x_0| \ge me^{-at}|x_0|$ *for all* $t \le 0$ *and* $x_0 \in \mathbb{R}^n$

Proof: We prove (c) $\Rightarrow$ (b) $\Rightarrow$ (a) $\Rightarrow$ (d) $\Rightarrow$ (c) $\Rightarrow$ (f) $\Rightarrow$ (e) $\Rightarrow$ (d). The proof is arranged to keep related parts together, and to leave the hardest part to the end.

(c) $\Rightarrow$ (b): If (c) holds then $\|e^{At}\| \le Me^{-at} \to 0$ as $t \to \infty$.

(b) $\Rightarrow$ (a): If (b) holds then $|e^{At}x_0| \le \|e^{At}\| \cdot |x_0| \to 0$ as $t \to \infty$.

(a) $\Rightarrow$ (d): Assume (d) does *not* hold. Then there exist a nonzero vector $z_0 \in \mathbb{C}^n$ and $\lambda \in \mathbb{C}$ with $Az_0 = \lambda z_0$ where $\lambda \in \mathbb{C}$ with $\operatorname{Re}\lambda \ge 0$. Then $e^{At}z_0 = e^{\lambda t}z_0 \not\to 0$ when $t \to \infty$. This non-convergence must hold for at least either the real part or the imaginary part of z_0, so (a) does not hold.

(e) $\Rightarrow$ (d): Assume (d) does *not* hold. Choose z_0 and λ as above. Then $|e^{At}z_0| = |e^{\lambda t}z_0| = e^{t\operatorname{Re}\lambda}|z_0| \not\to \infty$ as $t \to -\infty$, so (e) does not hold (again, split z_0 into real and imaginary parts to see this).

(f) $\Rightarrow$ (e): This is obvious.

(c) $\Rightarrow$ (f): Replace x_0 in (c) by $e^{-At}x_0$, to conclude that $|x_0| \le Me^{-at}|e^{-At}x_0|$ when $t \ge 0$. Now replace t with $-t$, and conclude $|x_0| \le Me^{at}|e^{At}x_0|$ when $t \le 0$. But this is exactly the inequality of (f), with $m = 1/M$.

(d) $\Rightarrow$ (c): Write $A = VJV^{-1}$ where J is the complex Jordan form of A, and furthermore write $J = M + N$ where M is a diagonal matrix whose diagonal entries are the same as those of J. The entries of N will be zero except possibly for some ones just above the diagonal. In particular N is nilpotent, i.e. $N^{q+1} = 0$ for some q. We may also note that $MN = NM$, so that $e^{Jt} = e^{Mt}e^{Nt}$.

Let β be the largest real part of any eigenvalue of A. By assumption, $\beta < 0$. The eigenvalues of A are also the diagonal entries of J, and therefore of M, and since M is diagonal, we get $\|e^{Mt}\| = e^{\beta t}$ when $t \ge 0$. Also

$$e^{Nt} = I + N + \frac{t^2}{2!}N^2 + \cdots + \frac{t^k}{k!}N^k$$

so we have $\|e^{Nt}\| \le p(t)$ for $t \ge 0$, where p is a polynomial. We end up, therefore, with

$$\begin{aligned}\|e^{At}\| &= \|Ve^{Jt}V^{-1}\| = \|Ve^{Mt}e^{Nt}V^{-1}\| \\ &\le \|V\| \cdot \|e^{Mt}\| \cdot \|e^{Nt}\| \cdot \|V^{-1}\| \le \|V\| \cdot \|V^{-1}\| \cdot e^{\beta t}p(t)\end{aligned}$$

from which we conclude

$$e^{at}\|e^{At}\| \le \|V\| \cdot \|V^{-1}\| \cdot e^{(a+\beta)t}p(t) \le M$$

for all $t \ge 0$, provided $a + \beta < 0$ and M is large enough. But this inequality is $\|e^{At}\| \le Me^{-at}$, which is (c). (We note that because $\beta < 0$, we can in fact satisfy both requirements $a > 0$ and $a + \beta < 0$ simultaneously.) ∎

The next theorem follows immediately from the above theorem by replacing A with $-A$:

Theorem 2 *Given a real $n \times n$ matrix A, the following are equivalent:*

(a) $\lim_{t\to-\infty} e^{At}x_0 = 0$ *for every* $x_0 \in \mathbb{R}^n$
(b) $\lim_{t\to-\infty} \|e^{At}\| = 0$
(c) *There exist constants* $a > 0$ *and* M *so that* $|e^{At}x_0| \le Me^{at}|x_0|$ *for all* $t \le 0$ *and* $x_0 \in \mathbb{R}^n$
(d) *Every eigenvalue of* A *har real part* > 0
(e) $\lim_{t\to\infty} |e^{At}x_0| = \infty$ *for every nonzero* $x_0 \in \mathbb{R}^n$

(f) *There exist constants* $a > 0$ *and* $m > 0$ *so that* $|e^{At}x_0| \geq me^{at}|x_0|$ *for all* $t \geq 0$ *and* $x_0 \in \mathbb{R}^n$

Remark. The theorems above are suitable replacements for Theorem 2 and 3 in section 1.9 of Perko's book, which are wrong as stated.

To see that Perko's Theorem 2 is wrong, consider the matrix

$$A = \begin{bmatrix} -1 & 0 \\ 0 & -2 \end{bmatrix}$$

which clearly satisfies (b).[1] We shall see that it does not satisfy (c): We find

$$e^{At} = \begin{bmatrix} e^{-t} & 0 \\ 0 & e^{-2t} \end{bmatrix}, \quad \|e^{At}\| = \max(e^{-t}, e^{-2t}) = \begin{cases} e^{-t} & t \geq 0, \\ e^{-2t} & t \leq 0. \end{cases}$$

But the righthand inequality of (c) states that $e^{ct}\|e^{At}\|$ is bounded, which means that $e^{(c-1)t}$ is bounded for $t > 0$, and $e^{(c-2)t}$ is bounded for $t < 0$. These statements imply both $c - 1 \leq 0$ and $c - 2 \geq 0$, which is impossible.

Reference. Lawrence Perko, *Differential equations and dynamical systems*, Second edition, Springer-Verlag 1996 (ISBN 0–387–94778–7) or First edition, Springer-Verlag 1991 (ISBN 0–387–96443–1)

[1] (b) and (c) here refer to Perko, of course.