Solution of problem 2.6: 1 and 2 in Perko

If you're reading this in maple, click on the icon in front of each subtitle to see the contents. Click again to hide it once more.

Some helpful functions needed in the solutions

First, we restart just to be safe. We need the linalg package:

```
> restart; with(linalg):
Warning, new definition for norm
Warning, new definition for trace
```

The gradient function is a bit like the grad function of the linalg package, but it lacks some of its features (including the error handling!), and it packages its output in a list, not a vector:

```
> gradient:=proc(f,U)
    local i,n;
    n:=nops(U);
    [seq(diff(f,U[i]),i=1..n)];
  end:
```

The following is for demonstration purposes only. It shows that gradient will work on a list of expressions, differentiating each element of the list with respect to each variable in the second list.

```
> gradient([x^2-y^2,x*y*z],[x,y,z]);
```

$$[[2\,x, y\,z], [-2\,y, x\,z], [0, x\,y]]$$

Now, we build a matrix from the above, transposing it in the process to conform with our usual conventions:

```
> DM:=proc(f,U)
    local m,n;
    m:=nops(f); n:=nops(U);
    transpose(matrix(n,m,gradient(f,U)));
  end:
> DM([x^2-y^2,x*y*z],[x,y,z]);
```

$$\begin{bmatrix} 2\,x & -2\,y & 0 \\ y\,z & x\,z & x\,y \end{bmatrix}$$

The above function generates the derivative of a vector-valued function of several variables from its expression. In other words, DM(f(x,y),[x,y]) is the matrix which we like to call Df(x,y) normally.

The function below will, given a set of substitutions s and an expression e, map the substitution over the expression. On most kinds of expression, subs will do that automatically, but if e is a matrix, the map has to be applied explicitly.

```
> mapsubs:=proc(s,e)
    local x;
    map(unapply('subs(s,x)',x),e);
  end:
```

2.6: 1(a)

First, define the righthand side of (a). I use the symbol a an index, to keep the various questions apart.

Next, we differentiate. diff(f[a],x1) would differentiate f[a] with respect to x1. To repeat this procedure for x1 and x2, we use the map procedure. Then we build a matrix from the result, and transpose it to fit with the conventions we use.
Then we find the zeros. solve wants a set of equations or expressions to make zero, but f[a] is a list. op(f[a]) pulls the elements out of the list, and {} makes a set from the result. Note that the answer is a sequence of sets of equations with the unknowns on the lefthand side.

```
> f[a]:=[x1-x1*x2, x2-x1^2];
  Df[a]:=DM(f[a],[x1,x2]);
  zeros[a]:=solve({op(f[a])},{x1,x2});
```

$$f_a := [x1 - x1\,x2,\, x2 - x1^2]$$

$$Df_a := \begin{bmatrix} 1 - x2 & -x1 \\ -2\,x1 & 1 \end{bmatrix}$$

$$zeros_a := \{x1 = 0, x2 = 0\}, \{x1 = 1, x2 = 1\}, \{x2 = 1, x1 = -1\}$$

First we consider the origin, which turns out to be an unstable node.

```
> op(1,[zeros[a]]); mapsubs(",Df[a]); eigenvals(");
```

$$\{x1 = 0, x2 = 0\}$$

$$\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

$$1, 1$$

Next we do (1,1) which turns out to be a saddle point.

```
> op(2,[zeros[a]]); mapsubs(",Df[a]); eigenvals(");
```

$$\{x1 = 1, x2 = 1\}$$

$$\begin{bmatrix} 0 & -1 \\ -2 & 1 \end{bmatrix}$$

$$-1, 2$$

(-1,1) is a saddle point as well:

```
> op(3,[zeros[a]]); mapsubs(",Df[a]); eigenvals(");
```

$$\{x2 = 1, x1 = -1\}$$

$$\begin{bmatrix} 0 & 1 \\ 2 & 1 \end{bmatrix}$$

$$-1, 2$$

2.6; 1(b)

```
> f[b]:=[-4*x2+2*x1*x2-8,4*x2^2-x1^2];
  Df[b]:=DM(f[b],[x1,x2]);
  zeros[b]:=solve({op(f[b])},{x1,x2});
```

$$f_b := [-4\,x2 + 2\,x1\,x2 - 8,\, 4\,x2^2 - x1^2]$$

$$Df_b := \begin{bmatrix} 2\,x2 & -4 + 2\,x1 \\ -2\,x1 & 8\,x2 \end{bmatrix}$$

$$zeros_b := \{x2 = \mathrm{RootOf}(\_Z + \_Z^2 + 2),\, x1 = -2\,\mathrm{RootOf}(\_Z + \_Z^2 + 2)\}, \{x2 = -1, x1 = -2\}, \{x2 = 2, x1 = 4\}$$

The first zero is not real, so we skip it. (RootOf is simply the zero(s) of the polynomial in the argument. Maple returns this when the exact root can not be determined.)

```
> op(2,[zeros[b]]); mapsubs(",Df[b]); eigenvals(");
```

$$\{x2 = -1, x1 = -2\}$$

$$\begin{bmatrix} -2 & -8 \\ 4 & -8 \end{bmatrix}$$

$$-5 + I\sqrt{23}, -5 - I\sqrt{23}$$

so this was a stable focus..

```
> op(3,[zeros[b]]); mapsubs(",Df[b]); eigenvals(");
```

$$\{x2 = 2, x1 = 4\}$$

$$\begin{bmatrix} 4 & 4 \\ -8 & 16 \end{bmatrix}$$

$$8, 12$$

and this is an ubstable node.

2.6: 1(c)

```
> f[c]:=[2*x1-2*x1*x2,2*x2-x1^2+x2^2];
  Df[c]:=DM(f[c],[x1,x2]);
  zeros[c]:=solve({op(f[c])},{x1,x2});
```

$$f_c := [2\,x1 - 2\,x1\,x2, 2\,x2 - x1^2 + x2^2]$$

$$Df_c := \begin{bmatrix} 2 - 2\,x2 & -2\,x1 \\ -2\,x1 & 2 + 2\,x2 \end{bmatrix}$$

$$zeros_c := \{x1 = 0, x2 = 0\}, \{x1 = 0, x2 = -2\}, \{x2 = 1, x1 = \mathrm{RootOf}(-3 + \_Z^2)\}$$

```
> op(1,[zeros[c]]); mapsubs(",Df[c]); eigenvals(");
```

$$\{x1 = 0, x2 = 0\}$$

$$\begin{bmatrix} 2 & 0 \\ 0 & 2 \end{bmatrix}$$

$$2, 2$$

that was an unstable node.

```
> op(2,[zeros[c]]); mapsubs(",Df[c]); eigenvals(");
```

$$\{x1 = 0, x2 = -2\}$$

$$\begin{bmatrix} 6 & 0 \\ 0 & -2 \end{bmatrix}$$

$$6, -2$$

which means a saddle point.

```
> op(3,[zeros[c]]); mapsubs(",Df[c]); eigenvals(");
```

$$\{x2 = 1, x1 = \mathrm{RootOf}(-3 + \_Z^2)\}$$

$$\begin{bmatrix} 0 & -2\,\mathrm{RootOf}(-3 + \_Z^2) \\ -2\,\mathrm{RootOf}(-3 + \_Z^2) & 4 \end{bmatrix}$$

$$-2, 6$$

another saddle point.

2.6: 1(d)

```
> f[d]:=[-x1,-x2+x1^2,x3+x1^2];
  Df[d]:=DM(f[d],[x1,x2,x3]);
  zeros[d]:=solve({op(f[d])},{x1,x2,x3});
```

$$f_d := [-x1, -x2 + x1^2, x3 + x1^2]$$

$$Df_d := \begin{bmatrix} -1 & 0 & 0 \\ 2\,x1 & -1 & 0 \\ 2\,x1 & 0 & 1 \end{bmatrix}$$

$$zeros_d := \{ x1 = 0, x2 = 0, x3 = 0 \}$$

```
> op(1,[zeros[d]]); mapsubs(",Df[d]); eigenvals(");
```

$$\{ x1 = 0, x2 = 0, x3 = 0 \}$$

$$\begin{bmatrix} -1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$-1, -1, 1$$

so, if we can trust Maple, only the origin is an equilibrium, and that is a saddle. There is a two-dimensional stable manifold and a one-dimensional unstable manifold.

2.6: 1(e)

```
> f[e]:=[x2-x1,k*x1-x2-x1*x3,x1*x2-x3];
  Df[e]:=DM(f[e],[x1,x2,x3]);
  zeros[e]:=solve({op(f[e])},{x1,x2,x3});
```

$$f_e := [x2 - x1, k\,x1 - x2 - x1\,x3, x1\,x2 - x3]$$

$$Df_e := \begin{bmatrix} -1 & 1 & 0 \\ k - x3 & -1 & -x1 \\ x2 & x1 & -1 \end{bmatrix}$$

$zeros_e := \{ x1 = 0, x2 = 0, x3 = 0 \},$

$$\{ x2 = \mathrm{RootOf}(-k + 1 + \_Z^2), x1 = \mathrm{RootOf}(-k + 1 + \_Z^2), x3 = k - 1 \}$$

```
> op(1,[zeros[e]]); mapsubs(",Df[e]); eigenvals(");
```

$$\{ x1 = 0, x2 = 0, x3 = 0 \}$$

$$\begin{bmatrix} -1 & 1 & 0 \\ k & -1 & 0 \\ 0 & 0 & -1 \end{bmatrix}$$

$$-1, -1 + \sqrt{k}, -1 - \sqrt{k}$$

so the origin is a sink if k<1 and a saddle if k>1.

```
> op(2,[zeros[e]]); mapsubs(",Df[e]); eigenvals(");
```

$$\{ x2 = \mathrm{RootOf}(-k + 1 + \_Z^2), x1 = \mathrm{RootOf}(-k + 1 + \_Z^2), x3 = k - 1 \}$$

$$\begin{bmatrix} -1 & 1 & 0 \\ 1 & -1 & -\mathrm{RootOf}(-k+1+\_Z^2) \\ \mathrm{RootOf}(-k+1+\_Z^2) & \mathrm{RootOf}(-k+1+\_Z^2) & -1 \end{bmatrix}$$

$$-2, \mathrm{RootOf}(\_Z^2 + \_Z + k - 1)$$

It's a good question whether Perko really intended us to find this zero, which is real when k>1. Note that the last eigenvalue is in fact two eigenvalues, so there are three eigenvalues altogether. The product of these two eigenvalues is k-1>0, so they both have the same sign if they are real (if not, they must be each other's complex conjugate). Their sum is -1, so this zero is always a sink.

2.6: 2

```
> f[x]:=[x2-x1,mu*x1-x2-x1*x3,x1*x2-x3];
  Df[x]:=DM(f[x],[x1,x2,x3]);
  zeros[x]:=solve({op(f[x])},{x1,x2,x3});
```

$$f_x := [x2 - x1, \mu\, x1 - x2 - x1\, x3, x1\, x2 - x3]$$

$$Df_x := \begin{bmatrix} -1 & 1 & 0 \\ \mu - x3 & -1 & -x1 \\ x2 & x1 & -1 \end{bmatrix}$$

$zeros_x := \{ x1 = 0, x2 = 0, x3 = 0 \},$

$$\{ x1 = \mathrm{RootOf}(-\mu + 1 + \_Z^2), x3 = -1 + \mu, x2 = \mathrm{RootOf}(-\mu + 1 + \_Z^2) \}$$

```
> op(1,[zeros[x]]); mapsubs(",Df[x]); eigenvals(");
```

$$\{ x1 = 0, x2 = 0, x3 = 0 \}$$

$$\begin{bmatrix} -1 & 1 & 0 \\ \mu & -1 & 0 \\ 0 & 0 & -1 \end{bmatrix}$$

$$-1, -1 + \sqrt{\mu}, -1 - \sqrt{\mu}$$

This is a sink when $\mu < 1$ and a saddle point when $1 < \mu$.

```
> op(2,[zeros[x]]); mapsubs(",Df[x]); eigenvals(");
```

$$\{ x1 = \mathrm{RootOf}(-\mu + 1 + \_Z^2), x3 = -1 + \mu, x2 = \mathrm{RootOf}(-\mu + 1 + \_Z^2) \}$$

$$\begin{bmatrix} -1 & 1 & 0 \\ 1 & -1 & -\mathrm{RootOf}(-\mu+1+\_Z^2) \\ \mathrm{RootOf}(-\mu+1+\_Z^2) & \mathrm{RootOf}(-\mu+1+\_Z^2) & -1 \end{bmatrix}$$

$$-2, \mathrm{RootOf}(\_Z^2 + \_Z - 1 + \mu)$$

The product of the last two eigenvalues (the RootOf expression) is $\mu - 1$ and their sum is -1. But these zeros are only real for $1 < \mu$, so the product is positive, and either both roots are negative, or they form a complex conjugate pair with a real part equal to $-\frac{1}{2}$. In either case, the equilibrium points are sinks.