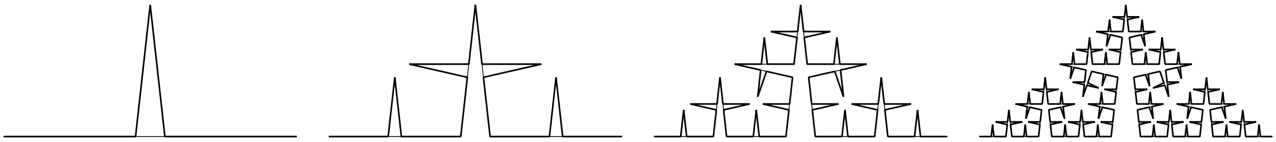
75045 Dynamical systems 1998

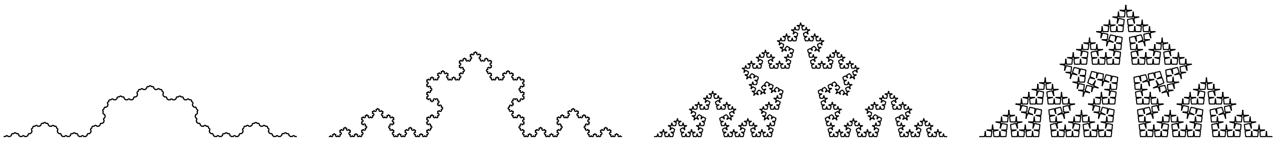
Problem set #1

#1–1: The unit circle in the complex plane is $\mathbb{T} = \{z \in \mathbb{C}\colon |z| = 1\}$. Show that the mapping $f\colon \mathbb{T} \to \mathbb{T}$ given by $f(z) = z^2$ is chaotic.

#1–2: (Exam, June 1997.) Let a be a given parameter, $\frac{1}{4} < a < \frac{1}{2}$. From a line segment of length L, remove a middle piece so that the two remaining pieces each have length aL, and two new line segments of length aL are added so that the four segments together form a connected curve as shown on the left.



Repeat this procedure on each of the four line segments, again on each of the resulting 16 segments, and so on, as suggested in the figure. In the limit we get a curve C_a (that is, there is a one-to-one continuous correspondence $f\colon [0,1] \to \mathbb{R}^2$ with $f([0,1]) = C_a$ – but you don't need to show this). What is the fractal dimension D_a of C_a?



The figure above shows approximations to C_a for $a = 0{,}28$, $a = 1/3$, $a = 0{,}4$ og $a = 0{,}45$. (Number two from the left, with $a = 1/3$, is the classic Koch curve.) What are the limits $\lim_{a\to 1/4} D_a$ and $\lim_{a\to 1/2} D_a$? Does this seem reasonable?

Problem set #2

From Perko: Section 1.2: 3, 6; section 1.3: 3, 4, 5, 8; section 1.5: 9.

Problem set #3

From Perko: Section 2.1: 1, 5; section 2.2: 1, 2; section 2.3: 2, 5, 6.

Problem set #1 – Solutions

#1–1: By an easy induction argument we get the formula $f^k(z) = z^{2^k}$ for the kth iteration of f.

To show that f is transitive we shall assume U and V are nonempty open subsets of $\mathbb{T}$, and must show the existence of some k so that $f^k(U) \cap V \neq \emptyset$. Equivalently, we must show that $U \cap f^{-k}(V) \neq \emptyset$.[1]

Since V is open, it contains an open arc: $V \supseteq J(a,b) = \{e^{i\theta}\colon a < \theta < b\}$ where $a < b$. Then $f^{-1}(V) \supseteq J(2a, 2b)$ and, by induction, $f^{-k}(V) \supseteq J(2^k a, 2^k b)$. Now, if k is large enough then $2^k b - 2^k a > 2\pi$, and then $f^{-k}(V) = \mathbb{T}$. But then $f^k(U) \cap V \neq \emptyset$ is an obvious conclusion.

To show that f is sensitively dependent on initial data, we must find an $\varepsilon > 0$ so that, for every $z \in \mathbb{T}$ and every neighbourhood V of z, there are $v, w \in V$ and some k so that $|f^k(v) - f^k(w)| > \varepsilon$.

Let $\varepsilon = 1/\sqrt{2}$. We shall show that, whenever $v \neq w$ and $|v - w| < \varepsilon$, there is some k so that $|f^k(v) - f^k(w)| > \varepsilon$. This is more than we need. Write $v/w = e^{ia}$ with $0 < |a| < \pi/2$. Let k be the smallest possible integer for which $2^k|a| > \pi/2$. Since $2^{k-1}|a| \leq \pi/2$, we therefore have $\pi/2 < 2^k|a| \leq \pi$, so that $|e^{i2^k a} - 1| > \varepsilon$. Thus $|f^k(v) - f^k(w)| = |v^{2^k} - w^{2^k}| = |(v/w)^{2^k} - 1| > \varepsilon$.

To show that the periodic points of f are dense in $\mathbb{T}$, simply note that z having period k means $f^k(z) = z$, or equivalently $z^{2^k} = z$. This means $z = e^{ia}$ with $2^k a = a + 2\pi n$. So the periodic points are those on the form e^{ia} with $a = 2\pi n/(2^k - 1)$, where k and n are integers with $k > 0$. Since such a are clearly dense in $\mathbb{R}$, the periodic points are dense in $\mathbb{T}$.

[1] In general, $g^{-1}(V) = \{z\colon f(z) \in V\}$, and $f^{-k}(V)$ is really an abbreviation of $(f^k)^{-1}(V)$.

#1–2: The set C_a is formed by four similitudes ψ_i $(i = 1, 2, 3, 4)$ which map the original line segment onto each of the four parts: The first is simply a division by a (if we put the origin at the left end of the original line segment), and all the others are such a division composed with a translation and maybe a rotation in the plane. The formula for the fractal dimension immediately yields

$$4a^{D_a} = 1 \quad \text{and hence } D_a = \frac{\ln 4}{\ln \dfrac{1}{a}}.$$

This requires a non-overlap condition. It is not hard to see that the whole of C_a must be inside a right triangle T with hypothenuse L, and the triangles $\psi_i(T)$ will not intersect other than at their courners – so that the non-overlap condition holds. (We cannot be more precise, given the required level of understanding for this course.)

We get

$$\lim_{a \to 1/4} D_a = 1 \quad \text{og} \quad \lim_{a \to 1/2} D_a = 2$$

which is reasonable, since C_a approaches the line segment L when $a \to \frac{1}{4}$, while it tends to fill the entire triangle T when $a \to \frac{1}{2}$.

Problem set #2 – Solutions

1.2: 3: (The numbers refer to Perko's book (first or second edition). So this is problem 3 in section 1.2.)

(a) We rewrite $\ddot{x} + \dot{x} = 2x = 0$ with $x_1 = x$ and $x_2 = \dot{x}$ to

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix} = \begin{bmatrix} x_2 \\ 2x_1 - x_2 \end{bmatrix} = \underbrace{\begin{bmatrix} 0 & 1 \\ 2 & -1 \end{bmatrix}}_{A} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$$

and find $\det(A - \lambda I) = \lambda^2 + \lambda - 2 = (\lambda + 1/2)^2 - 9/4 = (\lambda + 2)(\lambda - 1)$.

To proceed, we should, by the "standard recipe", find the eigenvectors corresponding to the two eigenvalues -2 and 1. But for a 2×2 matrix it is usually quicker to remember that we already know that $B = A - \lambda_1 I$ has rank 1 (unless it is zero), so B^2 must be a multiple of B (consider the null space and range of B and B^2).

In other words, consider

$$A - I = \begin{bmatrix} -1 & 1 \\ 2 & -2 \end{bmatrix}; \quad (A - I)^2 = \begin{bmatrix} 3 & -3 \\ -6 & 6 \end{bmatrix} = -3(A - I)$$

and write $A = I + B$ where $B^2 = -3B$, which yields $B^k = (-3)^{k-1}B$ for $k \geq 0$ (by induction — but note that the case $k = 0$ yields a different resultat $B^0 = I$). thus

$$e^{tA} = e^t e^{tB} = e^t\Big(I + \sum_{k=1}^{\infty} \frac{(-3)^{k-1}t^k}{k!}B\Big) = e^t\Big(I + \frac{e^{-3t} - 1}{-3}B\Big) = \frac{e^t}{3}\begin{bmatrix} 2 & 1 \\ 2 & 1 \end{bmatrix} + \frac{e^{-2t}}{3}\begin{bmatrix} 1 & -1 \\ -2 & 2 \end{bmatrix}.$$

The general solution is on the form $x(t) = \alpha e^t + \beta e^{-2t}$.

(b) $\ddot{x} + x = 0$ yields, as above with $x_1 = x$ and $x_2 = \dot{x}$,

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix} = \underbrace{\begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}}_{A} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$$

which is even simpler because $A^2 = -I$ so that

$$e^{tA} = \sum_{j=0}^{\infty} \frac{(-1)^j t^{2j}}{(2j)!} I + \sum_{j=0}^{\infty} \frac{(-1)^j t^{2j+1}}{(2j+1)!} A = (\cos t)I + (\sin t)A = \begin{bmatrix} \cos t & \sin t \\ -\sin t & \cos t \end{bmatrix}$$

and the general solution is of the form $x(t) = \alpha \cos t + \beta \sin t$, which we already knew of course.

(c) $\dddot{x} - 2\ddot{x} - \dot{x} + 2x = 0$ can be rewritten (as before) using $x_1 = x$, $x_2 = \dot{x}$ and $x_3 = \ddot{x}$ into

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \\ \dot{x}_3 \end{bmatrix} = \underbrace{\begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -2 & 1 & 2 \end{bmatrix}}_{A} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$$

from which we get $\det(A - \lambda I) = -(\lambda^3 - 2\lambda^2 - \lambda + 2) = -(\lambda^2 - 1)(\lambda - 2)$ and therefore eigenvalues ± 1 og -2. The general solution is on the form $x = \alpha e^t + \beta e^{-t} + \gamma e^{-2t}$. I am too lazy to determine the eigenvalues and hence get an expression for e^{tA}. It seems simpler to solve for α, β and γ given initial data $x_1(0) = x(0)$, $x_2(0) = \dot{x}(0)$ and $x_3(0) = \ddot{x}(0)$ and then get get an expression for e^{tA} from this?

I cannot quite resist the temptation to generalize a bit from the above three points: The equation $x^{(n)} + c_{n-1}x^{(n-1)} + \cdots + c_1\dot{x} + c_0 x = 0$ can be rewritten using $x_1 = x$, $x_2 = \dot{x}, \ldots, x_n = x^{(n-1)}$ to the form

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \\ \vdots \\ \dot{x}_{n-1} \\ \dot{x}_n \end{bmatrix} = \underbrace{\begin{bmatrix} 0 & 1 & 0 & \ldots & 0 \\ 0 & 0 & 1 & \ddots & 0 \\ \vdots & \vdots & \ddots & \ddots & \vdots \\ 0 & \ldots & \ldots & 0 & 1 \\ -c_0 & -c_1 & \ldots & -c_{n-2} & -c_{n-1} \end{bmatrix}}_{A} \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_{n-1} \\ x_n \end{bmatrix}$$

and it turns out that $\det(\lambda I - A = \lambda^n + c_{n-1}\lambda^{n-1} + \cdots + c_1\lambda + c_0$. You probably recongize here the characteristic polynomial of the original differential equation from the elementary theory for ordinary differential equations. In this sense, the theory of linear systems $\dot{x} = Ax$ is a fairly straightforward generalization.

1.2: 6: We have an explicit formula $\varphi(t, x_0) = e^{tA}x_0$ which is clearly a continuous function of x_0 (all linear mappings between spaces of the form $\mathbb{R}^n$ are continuous).

Note that if we draw the entire "forward trajectory" $\{\varphi(t, x_0) \colon t \geq 0\}$ and vary x_0, the picture does not have to change continuously with x_0, in spite of the contiuity we just proved. For a simple example, look at

$$A = \begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix}; \quad e^{tA} = \begin{bmatrix} e^{-t} & 0 \\ 0 & e^t \end{bmatrix}$$

and let $x_0 = (1, v)$. For $v > 0$ we get $x_2 \to \infty$ when $t \to \infty$, but for $v < 0$ we get $x_2 \to -\infty$, so that there is a sudden change in the *asymptotic behaviour* of the solution as x_0 passes the x_1 axis. (In both cases $x_1 \to 0$, and the trajectory follows the hyperbola $x_1 x_2 = v$.)

1.3: 3: The inequality $\|AB\| \leq \|A\|\|B\|$ applied to T and T^{-1} immediately yields $1 \leq \|T\|\|T^{-1}\|$, so we only need to divide by $\|T\|$ for the desired inequality.

1.3: 4: Write $S = I - T$. The assumption yields $\|S\| < 1$ and hence

$$\sum_{k=0}^{\infty} \|S^k\| \leq \sum_{k=0}^{\infty} \|S\|^k < \infty$$

and it follows that $R = \sum_{k=0}^{\infty} S^k$ converges absolutely. Multiplication by S yields

$$RS = \sum_{k=0}^{\infty} S^{k+1} = \sum_{k=1}^{\infty} S^k = R - I$$

so that $R(I - S) = I$, in other words $RT = I$.

1.3: 5: Question (a) is trivial, and I did the next three by the same trick as in **1.2: 3a** — that is, write $A = \lambda I + B$ where λ is an eigenvalue for A and use the fact that then $B^2 = \beta B$ for a suitable β, which I compute simply by squaring B. This yields

$$e^{tA} = e^{\lambda t} e^{tB} = e^{\lambda t}\Bigl(I + \sum_{k=1}^{\infty} \frac{\beta^{k-1} t^k}{k!} B\Bigr) = e^{\lambda t}\Bigl(I + \frac{e^{\beta t} - 1}{\beta} B\Bigr)$$

In the problem we use this only for $t = 1$, and I got these results:

$$\text{(a)}\ \begin{bmatrix} e^2 & 0 \\ 0 & e^{-3} \end{bmatrix};\quad \text{(b)}\ e\begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix} + e^{-1}\begin{bmatrix} 0 & -1 \\ 0 & 1 \end{bmatrix};\quad \text{(c)}\ e\begin{bmatrix} 1 & 0 \\ 5 & 0 \end{bmatrix};\quad \text{(d)}\ e^2\begin{bmatrix} 2 & -2 \\ 1 & -1 \end{bmatrix} + e^{-1}\begin{bmatrix} -1 & 2 \\ -1 & 2 \end{bmatrix}$$

(in (c) we got $\beta = 0$, which simply yields $e^B = I + B$). In (e) there are is a pair of non-real eigenvalues $2 \pm i$. Ths simplest thing is to write $A = 2I + B$ where $B^2 = -I$, and e^B can be found by substituting $t = -1$ in the solution of **1.2: 3b**. In (f) I use $A^2 = I$ to write

$$e^A = \sum_{j=0}^{\infty} \frac{1}{(2j)!} I + \sum_{j=0}^{\infty} \frac{1}{(2j+1)!} A = (\cosh 1) I + (\sinh 1) A$$

and we get the solutions

$$\text{(e)}\ e^2\begin{bmatrix} \cos 1 & -\sin 1 \\ \sin 1 & \cos 1 \end{bmatrix};\quad \text{(f)}\ \begin{bmatrix} \cosh 1 & \sinh 1 \\ \sinh 1 & \cosh 1 \end{bmatrix}$$

1.3: 8: We know that we must pick matrices that do not commute. The simplest example of non-commuting matrices I can think of is

$$A = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}, \quad B = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}$$

which yields

$$e^A = \begin{bmatrix} e & 0 \\ 0 & 1 \end{bmatrix}, \quad e^B = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}, \quad e^A e^B = \begin{bmatrix} e & e \\ 0 & 1 \end{bmatrix}, \quad e^{A+B} = \begin{bmatrix} e & e-1 \\ 0 & 1 \end{bmatrix}.$$

1.5: 9: I have this habit of doing things my own way, and rather than following the recipe in the problem, I prefer to differentiate the two equations $x_1 = r\cos\theta$ and $x_2 = r\sin\theta$ with respect to t, and get

$$\begin{bmatrix} \dot x_1 \\ \dot x_2 \end{bmatrix} = \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix}\begin{bmatrix} \dot r \\ r\dot\theta \end{bmatrix}$$

and therefore

$$\begin{bmatrix} \dot r \\ r\dot\theta \end{bmatrix} = \begin{bmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{bmatrix}\begin{bmatrix} \dot x_1 \\ \dot x_2 \end{bmatrix} = \frac{1}{r}\begin{bmatrix} x_1 & x_2 \\ -x_2 & x_1 \end{bmatrix}\begin{bmatrix} \dot x_1 \\ \dot x_2 \end{bmatrix}$$

which immediately yields the formulas for for $\dot r$ and $\dot\theta$ given in the problem. This yields til $\dot r = ar$ og $\dot\theta = b$ by a simple substitution. The solution with initial data $r(0) = r_0$ and $\theta(0) = \theta_0$ is $r(t) = r_0 e^{at}$ and $\theta(t) = \theta_0 + bt$.