75045 Dynamical systems 1998

Problem set #1(continued)

#1–3: (Exam, May 1992.)[1] Compute the fractal dimension of the set

$$\Big\{1, \frac{1}{2}, \frac{1}{3}, \dots, \frac{1}{n}, \dots, 0\Big\}.$$

Problem set #4

From Perko: Section 2.6: 1, 2; 2.7: 1, 2, 3.

Problem set #1 – Solutions (continued)

#1–3: We need to find upper and lower estimates on the number $N(\varepsilon)$ of intervals of length ε needed to cover the given set. We notice that the distance between two neighbours in the set is

$$\frac{1}{n} - \frac{1}{n+1} = \frac{1}{n(n+1)} > \frac{1}{(n+1)^2},$$

so an interval containing $1/n$ cannot contain any other points in the set if $n+1 < 1/\sqrt{\varepsilon}$. Clearly, then, we have $N(\varepsilon) > 1/\sqrt{\varepsilon}$.

We can easily cover the set with about twice that many intervals. For, after we have used $\lceil 1/\sqrt{\varepsilon}\rceil$ intervals to cover each of the points $1/n$ for $n = 1, 2, \dots, \lceil 1/\sqrt{\varepsilon}\rceil$, all the remaining points belong to the interval $[0, \sqrt{\varepsilon}]$ – which is easily covered using at most another $1/\sqrt{\varepsilon} + 1$ intervals. Thus $N(\varepsilon) \le 2/\sqrt{\varepsilon} + 1$.

The estimates arrived at in the previous two paragraphs can be summarized in

$$1 < N(\varepsilon)\sqrt{\varepsilon} \le 2 + \sqrt{\varepsilon}.$$

Taking logarithms, we find

$$0 < \ln N(\varepsilon) - \frac{1}{2}\ln\frac{1}{\varepsilon} < \ln 3$$

when $\varepsilon < 1$. Dividing by $\ln(1/\varepsilon)$ and letting $\varepsilon \to 0$ we finally get the fractal dimension:

$$D = \lim_{\varepsilon\to 0} \frac{\ln N(\varepsilon)}{\ln(1/\varepsilon)} = \frac{1}{2}.$$

It may be of interest to note that, in general, the estimate $a < \varepsilon^d N(\varepsilon) < b$ (when ε is small enough, and a, b, and d are constants) leads to the fractal dimension being d in the same way.

Problem set #3 – Solutions

2.1: 1: The problem contains two functions, both named f. I rename the second one to g to avoid confusion.

(a) The derivative of a function $f\colon \mathbb{R}^2 \to \mathbb{R}^2$ in a point x is really a linear map $\mathbb{R}^2 \to \mathbb{R}^2$ but, in practice, the most convenient representation of such a linear map is the corresponding matrix. The columns of this matrix are the partial derivatives of the function, which yields

$$\mathrm{D}f(x) = \begin{bmatrix} 1+x_2^2 & 2x_1x_2 \\ 2x_1 & -1+2x_2 \end{bmatrix}, \quad \mathrm{D}g(x) = \begin{bmatrix} 1+x_2^2+x_3^2 & 2x_1x_2 & 2x_1x_3 \\ -1+x_2x_3 & 1-x_3+x_1x_3 & -x_2+x_1x_2 \\ -2x_1 & 1 & 1 \end{bmatrix}$$

(b) The first component of $f(x) = 0$ is $x_1(1+x_2^2) = 0$ and hence $x_1 = 0$. Substituting this in the second component we get $-x_2 + x_2^2 = 0$, so $x_2 \in \{0, 1\}$. Substitute the answer from (a) to get:

$$\mathrm{D}f(0,0) = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}, \quad \mathrm{D}f(0,1) = \begin{bmatrix} 2 & 0 \\ 0 & 1 \end{bmatrix}$$

[1] I gave this problem, then forgot that I did so. Sorry about that.

(The point here, which is perhaps not so clear from the problem, is that the two equilibrium points are, respectively, a saddle point and an unstable node. By the Hartman–Grobman theorem, we know that the trajectories of $\dot{x} = f(x)$ resemble those of the linear system $\dot{x} = Ax$ med $A = Df(x_0)$ near any stationary point x_0 (i.e., where $f(x_0) = 0$.))

Similarly, $g(x) = 0$ yields $x_1(1 + x_2^2 + x_3^2) = 0$ for the first component, so that $x_1 = 0$. A substitution in the third component yields $x_2 + x_3 = 0$, and $x_1 = 0$, $x_3 = -x_2$ substituted into the second component yields $x_2 + x_2^2 = 0$ with the result $x_2 \in \{0, -1\}$. The derivatives become

$$\mathrm{D}g(0,0,0) = \begin{bmatrix} 1 & 0 & 0 \\ -1 & 1 & 0 \\ 0 & 1 & 1 \end{bmatrix}, \quad \mathrm{D}g(0,-1,1) = \begin{bmatrix} 3 & 0 & 0 \\ -2 & 0 & 1 \\ 0 & 1 & 1 \end{bmatrix}$$

(c) The second derivative is a bit more tricky. Since $\mathrm{D}f\colon \mathbb{R}^2 \to L(\mathbb{R}^2)$, $\mathrm{D}^2 f$ is a map $\mathrm{D}^2 f\colon \mathbb{R}^2 \to L(\mathbb{R}^2, L(\mathbb{R}^2))$. The notation is such that, if E and F are vector spaces, $L(E, F)$ is the space of linear maps $E \to F$, and $L(E)$ is shorthand for $L(E, E)$. In general, if $F\colon \mathbb{R}^2 \to E$ is a mapping into a (finite dimensional) vector space E, then $\mathrm{D}F\colon \mathbb{R}^2 \to L(\mathbb{R}^2, E)$. For every $x \in \mathbb{R}^2$ we know the linear map $\mathrm{D}F(x)$ if we know its action on the members of a basis, for example the natural basis $e_1 = (1, 0)$, $e_2 = (0, 1)$. But $\mathrm{D}F(x)e_j = \partial F/\partial x_j$. Applying this to $F = \mathrm{D}f$ we find

$$\mathrm{D}^2 f(x)e_1 = \begin{bmatrix} 0 & 2x_2 \\ 2 & 0 \end{bmatrix}, \quad \mathrm{D}^2 f(x)e_2 = \begin{bmatrix} 2x_2 & 2x_1 \\ 0 & 2 \end{bmatrix}$$

and therefore

$$\mathrm{D}^2 f(x)u = u_1 \begin{bmatrix} 0 & 2x_2 \\ 2 & 0 \end{bmatrix} + u_2 \begin{bmatrix} 2x_2 & 2x_1 \\ 0 & 2 \end{bmatrix}.$$

A different way to put it is as follows: A linear map $\varphi\colon \mathbb{R}^2 \to L(\mathbb{R}^2)$ yields a *bilinear*[2] map $\psi\colon \mathbb{R}^2 \times \mathbb{R}^2 \to \mathbb{R}^2$ by $\psi(u, v) = \varphi(u)(v)$, and every bilinear map ψ defines a linear map φ by the same formula. In mathematics it is common to *identify* isomorphic objects, that is, to pretend they are the same object.[3] So we consider $\mathrm{D}^2 f(x)$ as a bilinear as well as a linear map, and find

$$\begin{aligned} \mathrm{D}^2 f(x)(u, v) = \mathrm{D}^2 f(x)(u)(v) &= \left(u_1 \begin{bmatrix} 0 & 2x_2 \\ 2 & 0 \end{bmatrix} + u_2 \begin{bmatrix} 2x_2 & 2x_1 \\ 0 & 2 \end{bmatrix} \right) \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} \\ &= \begin{bmatrix} 2x_2 u_1 v_2 + 2x_2 u_2 v_1 + 2x_1 u_2 v_2 \\ 2u_1 v_1 + 2u_2 v_2 \end{bmatrix}. \end{aligned}$$

Note that this expression is symmetric: $\mathrm{D}^2 f(u, v) = \mathrm{D}^2 f(v, u)$. This is so in general when f is a C^2 function, and is nothing but the equality of mixed partial derivatives. (The problem asked for this answer only for $x_1 = 0$ and $x_2 = 1$.)

2.1: 5: $\dot{x} = 1/(2x)$ leads to $2x\,\mathrm{d}x = \mathrm{d}t$ which yields $x^2 = t - t_0$. If we substitute the initial data $x = 1$ when $t = 1$, we find $t_0 = 0$. From the initial data we also see that we must pick the positive square root, so $x(t) = \sqrt{t}$. Clearly, it follows that $\dot{x}(t) = 1/(2\sqrt{t})$ which approaches ∞ when $t \searrow 0$, in spite of the fact that x converges (to 0). It is the right hand side of the differential equation that «explodes» in the limit.

2.2: 1: A straightforward Picard iteration yields

$$\begin{aligned} &u_0(t) = 1, \quad u_1(t) = 1 + t, \quad u_2(t) = 1 + t + t^2 + \frac{1}{3}t^3, \\ &u_3(t) = 1 + t + t^2 + t^3 + \frac{2}{3}t^4 + \frac{1}{3}t^5 + \frac{1}{9}t^6 + \frac{1}{63}t^7, \quad \ldots \end{aligned}$$

We see a pattern here: u_k is a polynomial of degree $2^k - 1$ in which all the coefficients of terms of degree $\leq k$ are 1. It does not seem improbable that this should yield $\sum_{j=0}^{\infty} t^k = 1/(1-t)$ in the

[2] A bilinear map is defined by the requirements $\psi(au, v) = \psi(u, av) = a\psi(u, v)$, $\psi(u + v, w) = \psi(u, w) + \psi(v, w)$, $\psi(u, v + w) = \psi(u, v) + \psi(u, w)$ for all scalars a and vectors u, v and w.

[3] Clearly a dangerous practice if not exercised with due care, but very useful because it saves on notation and concepts.

limit, which solves the equation, at least for $|t| < 1$. A quick experiment using Maple (see the web page) points in the same direction.

2.2: 2: Picard iteration for $\dot{x} = Ax$ with $x(0) = x_0$ using $u_0(t) = x_0$ yields:

$$u_1(t) = x_0 + \int_0^t Ax_0 \,\mathrm{d}s = (I + tA)x_0,$$
$$u_2(t) = x_0 + \int_0^t A(I + sA)x_0 \,\mathrm{d}s = \Bigl(I + tA + \frac{t^2}{2}A^2\Bigr)x_0$$

and it is not unreasonable to guess that

$$u_k(t) = \sum_{j=0}^{k} \frac{t^j}{j!}A^k$$

which is easily shown by induction. But this is the k–th partial sum of the sequence for $e^{tA}x_0$, and converges to this for all t.

2.3:2:

(a) We solve $\dot{x} = f(x)$ with initial data $x(0) = y$. So we should have $\dot{x}_1 = -x_1$, which yields $x_1 = y_1e^{-t}$ with the given initial data. Next $\dot{x}_2 = -x_2 + x_1^2 = -x_2 + y_1^2e^{-2t}$, with general solution $x_2 = Ae^{-t} - y_1^2e^{-2t}$. The initial data yield $A = y_2 + y_1^2$. Lastly, $\dot{x}_3 = x_3 + x_1^2 = x_3 + y_1^2e^{-2t}$ with general solution $x_3 = Be^t - (y_1^2/3)e^{-2t}$. The initial data yield $B = y_3 + y_1^2/3$. So we end up with

$$u(t, y) = \Bigl[y_1e^{-t}, \quad (y_2 + y_1^2)e^{-t} - y_1^2e^{-2t}, \quad \Bigl(y_3 + \frac{y_1^2}{3}\Bigr)e^t - \frac{y_1^2}{3}e^{-2t}\Bigr]^{\mathrm{T}}$$

and this yields

$$\Phi(t, y) = \frac{\partial u}{\partial y} = \begin{bmatrix} e^{-t} & 0 & 0 \\ -2y_1e^{-2t} + 2y_1e^{-t} & e^{-t} & 0 \\ -\frac{2y_1}{3}e^{-2t} + \frac{2y_1}{3}e^t & 0 & e^t \end{bmatrix}$$

and hence

$$\dot{\Phi} = \begin{bmatrix} -e^{-t} & 0 & 0 \\ 4y_1e^{-2t} - 2y_1e^{-t} & e^{-t} & 0 \\ \frac{4y_1}{3}e^{-2t} + \frac{2y_1}{3}e^t & 0 & e^t \end{bmatrix}.$$

Further,

$$Df(x) = \begin{bmatrix} -1 & 0 & 0 \\ 2x_1 & -1 & 0 \\ 2x_1 & 0 & 1 \end{bmatrix}$$

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$$Df\bigl(u(t, y)\bigr)\Phi = \begin{bmatrix} -1 & 0 & 0 \\ 2y_1e^{-t} & -1 & 0 \\ 2y_1e^{-t} & 0 & 1 \end{bmatrix} \begin{bmatrix} e^{-t} & 0 & 0 \\ -2y_1e^{-2t} + 2y_1e^{-t} & e^{-t} & 0 \\ -\frac{2y_1}{3}e^{-2t} + \frac{2y_1}{3}e^t & 0 & e^t \end{bmatrix} = \Phi(t, y)$$

after a bit of work.

(b) Same procedure as previously. We get $\dot{x}_1 = x_1^2$ which, after solving and substituting $x_1(0) = y_1$, yields $x_1 = y_1/(1 - y_1t)$. Further, $\dot{x}_2 = x_2 + x_1^{-1} = x_2 + y_1^{-1} - t$ which is perhaps most easily solved using the integrating factor e^{-t}, and yields $x_2 = t + 1 - y_1^{-1} + (y_2 - 1 + y_1^{-1})e^t$. Summed up:

$$u(t, x) = \begin{bmatrix} \dfrac{y_1}{1 - y_1t} \\ t + 1 - \dfrac{1}{y_1} + \Bigl(y_2 - 1 + \dfrac{1}{y_1}\Bigr)e^t \end{bmatrix}, \quad \Phi = \frac{\partial u}{\partial y} = \begin{bmatrix} \dfrac{1}{(1 - y_1t)^2} & 0 \\ \dfrac{1 - e^t}{y_1^2} & e^t \end{bmatrix}$$

Furthermore, we find

$$Df(x) = \begin{bmatrix} 2x_1 & 0 \\ -\dfrac{1}{x_1^2} & 1 \end{bmatrix}, \quad \dot{\Phi} = \begin{bmatrix} \dfrac{2y_1}{(1 - y_1t)^3} & 0 \\ -\dfrac{e^t}{y_1^2} & e^t \end{bmatrix}$$

and finally

$$Df\big(u(t,x)\big)\Phi = \begin{bmatrix} \dfrac{2y_1}{1-y_1t} & 0 \\ -\dfrac{(1-y_1t)^2}{y_1^2} & 1 \end{bmatrix} \begin{bmatrix} \dfrac{1}{(1-y_1t)^2} & 0 \\ \dfrac{1-e^t}{y_1^2} & e^t \end{bmatrix} = \dot\Phi.$$

2.3: 5: Liouville's theorem, which is mentioned in the problem, can be written as

$$\frac{\mathrm{d}}{\mathrm{d}t} \ln\det\Phi(t) = \operatorname{tr}\big(\dot\Phi(t)\Phi(t)^{-1}\big)$$

as long as Φ is invertible.[4] If $\dot\Phi = A(t)\Phi$ the right hand side becomes $\operatorname{tr} A(t)$. If, additionally, $\Phi(0) = I$ this can be integrated to $\ln\det\Phi(t) = \int_0^t \operatorname{tr} A(s)\,\mathrm{d}s$, and so we get

$$\det\Phi = \exp\int_0^t \operatorname{tr} A(s)\,\mathrm{d}s$$

as the problem claims.

I prefer to reformulate the problem in terms of the flow: $\varphi_t(x_0) = u(t,x_0)$ is the solution of $\dot x = f(x)$ with initial value x_0 at time t. I use the notation $\mathrm{D}\varphi_t$ for the derivative of the function φ_t, which the problem calls $\partial u(t,y)/\partial y$. We know it satisfies the equation

$$\frac{\mathrm{d}}{\mathrm{d}t}\mathrm{D}\varphi_t(x_0) = \mathrm{D}f\big(\varphi_t(x_0)\big)\mathrm{D}\varphi_t(x_0)$$

which is precisely of the form $\dot\Phi = A(t)\Phi$, and Liouville's theorem leads to

$$\det\mathrm{D}\varphi_t(x_0) = \exp\int_0^t \operatorname{tr}\mathrm{D}f\big(\varphi_s(x_0)\big)\,\mathrm{d}s = \exp\int_0^t \operatorname{div} f\big(\varphi_s(x_0)\big)\,\mathrm{d}s.$$

$\det\mathrm{D}\varphi_t(x_0)$ measures how a small volume near x_0 is changed as it moves with the flow over a time t (this is the change of variables formula for multiple integrals). When the determinant is greater than 1, the volume is expanding by a corresponding factor, and when it is less that 1, the volume shrinks. We see now that it is the sign of $\operatorname{div} f$ along the trajectory from x_0 to $\varphi_t(x_0)$ that decides which of the two cases will happen. It is this phenomenon that gives rise to the name "divergence".

2.3: 6: This is a rather direct consequence of the previous problem, not least the comment at the end. Let $R_1 = \varphi_t(R_0)$. Since φ_t is a diffeomorphism (it is differentiable with a differentiable inverse), we may change variables in the integral for the volume of R_1:

$$V(R_1) = \int_{R_1} \mathrm{d}y$$

(where this is short for a multiple integral and $\mathrm{d}y$ is short for $\mathrm{d}y_1\cdots\mathrm{d}y_n$) by putting $y = \varphi_t(x)$ and letting x vary over R_0, with the result

$$V(R_1) = \int_{R_0} |\det\mathrm{D}\varphi_t(x)|\,\mathrm{d}x.$$

But if $\operatorname{div} f = 0$ everywhere, the formula of the previous problem shows that $\det\mathrm{D}\varphi_t(x) = 1$ everywhere, and the integrand above becomes 1 – and, therefore, $V(R_1) = V(R_0)$. In other words, the flow is volume preserving if $\operatorname{div} f = 0$.

Conversely, if $\operatorname{div} f \neq 0$ in at least on point, we can find an open, nonempty set U_0 in which $\operatorname{div} f > 0$ (or one where $\operatorname{div} f < 0$, but this is treated the same way). Let R_0 be an open, nonempty subset of U_0 so that $\overline{R}_0$ is compact and contained in U_0. Then there exists some $\delta > 0$ so that $\varphi_t(R_0) \subseteq U_0$ whenever $|t| < \delta$. For such a $t > 0$ we find, using the formula of the previous problem, that $\det\mathrm{D}\varphi_t > 1$ in R_0, and the integral formula of the previous paragraph thus yields $V(R_1) > V(R_0)$ so that φ_t is not volume preserving.

[4] I wrote a brief derivation of this result. It was handed out earlier.