75045 Dynamical systems 1998

Problem set #5

From Perko: Section 2.9: 4, 5, 7, 8.

Problem set #6

#6: 1: Assume H is a real-valued C^1 function so that $DH(x)f(x) = g(x)H(x)$ where g is a real, continuous function. Show that then the point set given by $H(x) = 0$ is invariant for the system $\dot{x} = f(x)$. (Hint: Use Grönwall.)

#6: 2: Prove theorem 6 in Perko section 2.10.

From Perko: Section 2.12: 1–4.

Problem set #4 — Solutions

2.6: 1, 2: I did this in Maple. See the attached Maple worksheet (also available from the Web page).

2.7: 1: Write the system as

$$\begin{bmatrix}\dot{x}_1\\ \dot{x}_2\end{bmatrix} = \underbrace{\begin{bmatrix}1 & 6\\ 4 & 3\end{bmatrix}}_{A}\begin{bmatrix}x_1\\ x_2\end{bmatrix} + \underbrace{\begin{bmatrix}x_1x_2\\ -x_1^2\end{bmatrix}}_{F(x)}$$

where $F(x)$ is quadratic in (x_1, x_2). We discover that A has the eigenvalues -3 and 7 with eigenvectors $(3,-2)^{\mathrm{T}}$ and $(1,1)^{\mathrm{T}}$ respectively, so that $AV = VB$ with

$$B = \begin{bmatrix}-3 & 0\\ 0 & 7\end{bmatrix}, \quad V = \begin{bmatrix}3 & 1\\ -2 & 1\end{bmatrix}, \quad V^{-1} = \frac{1}{5}\begin{bmatrix}1 & -1\\ 2 & 3\end{bmatrix}.$$

Our system $\dot{x} = Ax + F(x)$ is rewritten using $x = Vy$ to $V\dot{y} = AVy + F(Vy) = VBy + F(Vy)$, i.e., to $\dot{y} = By + G(y)$ where

$$G(y) = V^{-1}F(Vy) = \frac{1}{5}\begin{bmatrix}1 & -1\\ 2 & 3\end{bmatrix} F\left(\begin{bmatrix}3y_1 + y_2\\ -2y_1 + y_2\end{bmatrix}\right) = \frac{1}{5}\begin{bmatrix}3y_1^2 + 7y_1y_2 + 2y_2^2\\ -39y_1^2 - 16y_1y_2 - y_2^2\end{bmatrix}$$

(with a little help from Maple at the end).

2.7: 2: If we follow the existence proof for the stable manifold, we should have $u^{(0)}(t,a) = 0$, while

$$u^{(j+1)}(t,a) = U(t)a + \int_0^t U(t-s)G(u^{(j)}(s,a))\,\mathrm{d}s - \int_t^\infty V(t-s)G(u^{(j)}(s,a))\,\mathrm{d}s.$$

In this case we get

$$B = \begin{bmatrix}-1 & 0\\ 0 & 1\end{bmatrix}, \quad G(x) = \begin{bmatrix}0\\ x_1^2\end{bmatrix}, \quad U(t) = \begin{bmatrix}e^{-t} & 0\\ 0 & 0\end{bmatrix}, \quad V(t) = \begin{bmatrix}0 & 0\\ 0 & e^t\end{bmatrix}.$$

But then both of the integrals in the above recursion formula have zero in their first components, so that

$$u_1^{(j+1)}(t,a) = a_1e^{-t}.$$

The second component of the recursion formula yields

$$u_2^{(j+1)}(t,a) = -\int_t^\infty e^{t-s}\big(u_1^{(j)}(s,a)\big)^2\,\mathrm{d}s.$$

For $j = 0$ this yields $u_2^{(1)}(t,a) = 0$, while for $j \geq 1$ we get

$$u_2^{(j+1)}(t,a) = -\int_t^\infty e^{t-s}\big(a_1e^{-s}\big)^2\,\mathrm{d}s = -\frac{a_1^2}{3}e^{-2t}.$$

To sum it up: $u^{(j)}(t,a)$ is independent of j for $j \geq 2$, and therefore converges trivially to the limit

$$u(t,a) = \begin{bmatrix} a_1 e^{-t} \\ -\frac{a_1^2}{3} e^{-2t} \end{bmatrix} \quad \text{with } u(0,a) = \begin{bmatrix} a_1 \\ -\frac{a_1^2}{3} \end{bmatrix}$$

as $j \to \infty$. However it is precisely $u(0,a)$ which parameterizes the local stable manifold S, which thus has the equation $y_2 = -y_1^2/3$.

2.7: 3: $\dot{x}_1 = -x_1$ has the general solution $x_1 = Ae^{-t}$. We can substitute this into the second equation $\dot{x}_2 = x_2 + x_1^2$ and get $\dot{x}_2 = x_2 + A^2 e^{-2t}$, which has the general solution $x_2 = Be^t - (A^2/4)e^{-2t}$.

It is quite clear that the x_2 axis ($x_1 = 0$) is invariant. From the previous question we know that the curve given by $x_2 + x_1^2/3 = 0$ should also be invariant, but it is not a bad idea to check this explicitly:

$$\frac{\mathrm{d}}{\mathrm{d}t}\Big(x_2 + \frac{x_1^2}{3}\Big) = \dot{x}_2 + \frac{2}{3}x_1\dot{x}_1 = x_2 + x_1^2 - \frac{2}{3}x_1^2 = x_2 + \frac{x_1^2}{3}$$

so that $u(t) = x_2 + x_1^2/3$ satisfies $\dot{u} = u$. But we know that in this case $u(t) = 0$ for all t provided $u(t) = 0$ for some t.

Problem set #5 – Solutions

2.9: 4: We can do the questions (a) and (b) in one go by writing

$$\begin{aligned} \dot{x}_1 &= -x_2 + \mu(x_1^2 + x_2^2)x_1, \\ \dot{x}_2 &= x_1 + \mu(x_1^2 + x_2^2)x_2 \end{aligned}$$

where $\mu = -1$ in (a) and $\mu = 1$ in (b). If we try the Liapunov function $V = x_1^2 + x_2^2$ we quickly find

$$\dot{V} = 2(x_1\dot{x}_1 + x_2\dot{x}_2) = 2(-x_1x_2 + \mu(x_1^2 + x_2^2)x_1^2 + x_1x_2\mu(x_1^2 + x_2^2)x_2^2) = 2\mu(x_1^2 + x_2^2)^2 \neq 0$$

with the same sign as μ for $(x_1, x_2) \neq 0$. Thus we have asymptotic stability in (a) and instability in (b).

(c) Here we have

$$\begin{aligned} \dot{x}_1 &= -x_2 - x_1x_2, \\ \dot{x}_2 &= x_1 + x_1^2 \end{aligned}$$

which immediately yields

$$\dot{V} = 2(x_1\dot{x}_1 + x_2\dot{x}_2) = 2(-x_1x_2 - x_1^2x_2 + x_1x_2 + x_1^2x_2) = 0.$$

This gives us stability, but not asymptotic stability (the trajectories lie on circles with their centers at the origin). In fact, we can write

$$\dot{x}_1 = -(1 + x_1)x_2, \quad \dot{x}_2 = (1 + x_1)x_1$$

which apart from the common factor $1 + x_1$ is the same system as $(\dot{x}_1, \dot{x}_2) = (-x_2, x_1)$ with solution $(x_1, x_2) = \big(A\cos(t - a), A\sin(t - a)\big)$. The vector $1 + x$ means that the solutions for the two systems follow the same trajectories, but with different speeds (and opposite directions, when $x_1 < -1$).

2.9: 5: All the systems in this question have an equilibrium at the origin. In the questions (a–c) this equilibrium is hyperbolic, so there is hardly any need for a Liapunov function to decide stability! In questions (d), on the other hand, the equilibrium is non-hyperbolic: The linearized system has a center. I haven't found a Liapunov function for this, so we'll need to get back to this problem later.

2.9: 7: Here f is an even polynomial and g an odd one, and

$$F(x) = \int_0^x f(s)\,\mathrm{d}s, \quad G(x) = \int_0^x g(s)\,\mathrm{d}s.$$

In particular, F is an odd polynomial and G an even polynomial with $F(0) = G(0) = 0$, $F' = f$, and $G' = g$. Since we shall rewrite the equation

$$\ddot{x} + f(x)\dot{x} + g(x) = 0$$

to a system containing the equation $\dot{x}_2 - g(x_1)$ it is clear that we ought to put $x_1 = x$ and try to write $\ddot{x} + f(x)\dot{x}$ as derivative. But that is simple, since

$$\frac{\mathrm{d}}{\mathrm{d}t}F(x) = F'(x)\dot{x} = f(x)\dot{x}.$$

So we only need to define $x_2 = \dot{x}_1 + F(x_1)$ to obtain the desired equation $\dot{x}_2 = \ddot{x}_1 + f(x_1)\dot{x}_1 = -g(x_1)$ from the given equation. If we clean up, we get the Lienard system

$$\begin{aligned} \dot{x}_1 &= x_2 - F(x_1), \\ \dot{x}_2 &= -g(x_1). \end{aligned}$$

The origin is an equilibrium because $F(0) = 0$ and $g(0) = 0$ (since g is odd).

The given data seems to point us in the direction of looking for a Liapunov function. It is very tempting to let $G(x_1)$ be a part of it, because we are given the information that $G(x_1) > 0$ when $x_1 \neq 0$ is small enough. We find

$$\frac{\mathrm{d}}{\mathrm{d}t}G(x_1) = g(x_1)\dot{x}_1 = g(x_1)x_2 - g(x_1)F(x_1)$$

where we can control the last term, but not the first. But we have

$$\frac{\mathrm{d}}{\mathrm{d}t}\frac{x_2^2}{2} = x_2\dot{x}_2 = -x_2 g(x_1)$$

and so we see that $V = G(x_1) + x_2^2/2$ should be a usable Liapunov function with $\dot{V} = -g(x_1)F(x_1)$. This has the advantage that $V > 0$ when $(x_1, x_2) \neq (0,0)$ is small enough, and so we have $\dot{V} = -g(x_1)F(x_1)$ which yields precisely the stability properties we were asked about, except we only get $\dot{V} \overset{\geq}{<}$, while $\dot{V} \gtrless 0$ only when $x_1 \neq 0$ ($\overset{\geq}{<}$ stands for either $\geq$ or $\leq$, and $\gtrless$ for either $>$ or $<$, depending on the sign of μ). That this still gives asymptotic (in)stability needs a separate argument.

2.9: 8: van der Pol's equation $\ddot{x} + \varepsilon(x^2 - 1)\dot{x} + x = 0$ is precisely an equation of this type, with $f(x) = \varepsilon(x^2 - 1)$ and $g(x) = x$. This yields $G(x) = x^2/2 > 0$ for $x \neq 0$ and $F(x) = \varepsilon(x^3/3 - x)$ so that $g(x)F(x) = \varepsilon(x^2/3 - 1)x^2 \neq 0$ with the opposite sign of ε, when $0 < |x| < \sqrt{3}$. For $\varepsilon > 0$ the origin is therefore and unstable equilibrium, and for $\varepsilon < 0$ it is stable.