75045 Dynamical systems 1998

Final problem set: #7

From Perko: Section 3.2: 1, 4; 3.3: 2; 3.4: 1; 3.12: 7,9

Problem set #5 — Solutions (continued)

2.9: 5(d): After much searching, I still haven't found a Liapunov function for this system! However, one can use the symmetry of the given system

$$\begin{aligned}\dot{x} &= -4y + x^2 \\ \dot{y} &= 4x + y^2\end{aligned}$$

to show that the system has closed orbits. In fact, if $\big(x(t), y(t)\big)$ solves this system then so does $\big(-y(-t), -x(-t)\big)$. Consider now a solution starting at the diagonal $x + y = 0$: If $x(0) + y(0) = 0$ then $\big(-y(-t), -x(-t)\big) = \big(x(t), y(t)\big)$ for $t = 0$, and so this relation holds for all t by uniqueness.

Now for small x and y we find in polar coordinates $\dot{\theta} \approx 4$ and $\dot{r} \approx 0$, so certainly the trajectory will meet the diagonal $x + y = 0$ again for some positive time $t_1 \approx \pi/4$. Then $x(t_1) = -y(t_1) = x(-t_1)$, and similarly $y(t_1) = y(-t_1)$. Thus the trajectory is a cycle with period $2t_1$. It follows that the origin is a stable, but not asymptotically stable, equilibrium for this system.

(Stability follows because any trajectory starting inside a small cycle around the origin must stay inside. Clearly, this argument, which relies on the Jordan curve theorem, would not work in higher dimensions than two without some further work.)

Problem set #6 — Solutions

#6: 1: We must interpret the given relation $\mathrm{D}H(x)f(x) = g(x)H(x)$ correctly: Here g og H are scalar functions, while f is vector valued. Since the righthand side is scalar, so must the lefthand side be. And this is indeed so, for $\mathrm{D}H(x)$ is a linear function from $\mathbb{R}^n$ to $\mathbb{R}$ in the abstract, or a $1 \times n$ matrix (row vector) in the concrete version.

We must show that $H(x) = 0$ is invariant for the system $\dot{x} = f(x)$, i.e., that if $H(x(0)) = 0$ for a solution x, then $H(x(t)) = 0$ for all t. Differentiation with respect to t and the chain rule yield

$$\dot{H} = \frac{\mathrm{d}}{\mathrm{d}t}H(x(t)) = \mathrm{D}H(x(t))\dot{x}(t) = \mathrm{D}H(x(t))f(x(t)) = g(x)H(x).$$

Thus $|\dot{H}| \le M|H(x(t))|$ where M is such that $|g(x(t))| \le M$ for $0 \le t \le T$, where we have chosen an interval $[0, T]$ to work in. Hence, since $H(x(0)) = 0$,

$$|H(x(t))| = \left|\int_0^t \dot{H}(x(s))\,\mathrm{d}s\right| \le M\int_0^t |H(x(s))|\,\mathrm{d}s$$

from which Grönwall's inequality immediately yields $H(x(t)) = 0$ for $0 \le t \le T$. Since $T > 0$ was arbitrary, this holds for all $t > 0$ as long as the solution $x(t)$ is defined. The case $t < 0$ is treated the same way, or you just need to remember that $x(-t)$ solves $\dot{x} = -f(x)$ and just us what we showed for $t > 0$.

#6: 2: (In this solution the kernel of the argument is quite short, and comes at the end. I'll use most of the time showing that the orbits near the origin actually must go around the origin, or more precily that they satisfy $|\dot{\theta}| > c$ for a positive constant c.)

Here we consider the system

$$\dot{x} = P(x,y), \quad \dot{y} = Q(x,y).$$

If we replace (t,x,y) by $(-t,x,-y)$ this becomes $-\dot{x} = P(x,-y)$ and $\dot{y} = Q(x,-y)$, and invarians under this transformation (symmetry wrt.þhe x axis) means

$$P(x,-y) = -P(x,y), \quad Q(x,-y) = Q(x,y).$$

If you draw a picture, you will find that this means the vector field $f = (P,Q)$ is mapped to $-f$ by taking the mirror image about the x axis. A solution curve mirrored about the x axis thus becomes a new solution curve, but with time progressing in the opposite direction.

If we differentiate the symmetry relations above with respect to x and y, respectivly and put $y = 0$, we get $P_x(x,0) = -P_x(x,y)$ and $-Q_y(x,0) = Q_y(x,0)$, so that $P_x(x,0) = Q_y(x,0) = 0$. In particular, for the matrix $A = \mathrm{D}f(0,0)$ we fine

$$A = \begin{bmatrix} P_x(0,0) & P_y(0,0) \\ Q_x(0,0) & Q_y(0,0) \end{bmatrix} = \begin{bmatrix} 0 & a \\ b & 0 \end{bmatrix} \quad \text{der } a = P_y(0,0) \text{ og } b = Q_x(0,0).$$

The linearized system, with coefficient matrix A, has eigenvalues given by the characteristic equation $\det(A - \lambda I) = 0$, that is $\lambda^2 - ab = 0$. These become pure imaginary, so that the linearized system has a center if and only if $ab < 0$. For simplicity I will assume $a < 0$ and $b > 0$ from now on.

Polar coordinates give us

$$\dot{\theta} = \frac{x\dot{y} - y\dot{x}}{r^2} = \frac{xQ(x,y) - yP(x,y)}{r^2} = \frac{x(bx + o(r)) - y(ay + o(r))}{r^2} = \frac{bx^2 - ay^2}{r^2} + o(1)$$

and our assumptions on a and b give $bx^2 - ay^2 \geq r^2 \min(|b|,|a|)$ so that we can find constants $c, \delta > 0$ with $\dot{\theta} > c$ if $r < \delta$. It follows that an orbit starting sufficiently close to the origin will do a half revolution around the origin in time $< \pi/c$, and in particular such a point on the positive x axis will move to the negative x axis. Thus, if $x(0) > 0$ is small enough and $y(0) = 0$, a solution (x,y) will satisfy $x(\tau) < 0$ and $y(\tau) = 0$ for some $\tau > 0$, where $y(t) > 0$ if $0 < t < \tau$.

By the symmetry $(x(-t), -y(-t))$ will solve the same problem, so that $x(-t) = x(t)$ og $y(-t) = -y(t)$ for all t by uniqueness. In particular $(x(-\tau), y(-\tau)) = (x(\tau), y(\tau))$, and (x,y) is periodic with period 2τ. This shows that the origin is a center for the system.

2.12: 1:

(a) We need to find a function H with

$$\frac{\partial H}{\partial y} = a_{11}x + a_{12}y + Ax^2 - 2Bxy + Cy^2,$$
$$\frac{\partial H}{\partial x} = -a_{21}x + a_{11}y^2 - Dx^2 + 2Axy - By^2.$$

Integrate the first relation with respect to y and the other with respect to x, and get

$$H = a_{11}xy + \frac{a_{12}}{2}y^2 + Ax^2y - Bxy^2 + \frac{C}{3}y^3 + F(x),$$
$$H = -\frac{a_{21}}{2}x^2 + a_{11}xy^2 - \frac{D}{3}x^3 + Ax^2y - Bxy^2 + G(y)$$

for suitable functions F og G. We see that both of the formulas for H hold if we put

$$F(x) = -\frac{a_{21}}{2}x^2 - \frac{D}{3}x^3, \quad G(y) = \frac{a_{12}}{2}y^2 + \frac{C}{3}y^3$$

which yields

$$H(x,y) = -\frac{a_{21}}{2}x^2 + a_{11}xy + \frac{a_{12}}{2}y^2 - \frac{D}{3}x^3 + Ax^2y - Bxy^2 + \frac{C}{3}y^3.$$

(b) If the system is Hamiltonian and we write $f = (f_1, f_2)$ then $f_1 = H_y$ and $f_2 = -H_x$ (where an index denotes a partial derivative), and hence $\operatorname{div} f = f_{1x} + f_{2y} = H_{yx} - H_{xy} = 0$, at least as long as $H \in C^2$, since mixed partial derivatives are then equal.

The converse holds in general only in a simply connected region: Given f with $\operatorname{div} f = 0$ we seek a function H so that $\nabla H = g = (-f_2, f_1)$. Here the vector field g will satisfy $g_{1y} = g_{2x}$ (since $g_{2x} - g_{1y} = f_{1x} + f_{2y} = 0$), and according to Green's theorem the line integral $\int_a^b g(z) \cdot \mathrm{d}z$ will be independent of the path so long as we stay within a simply connected region. Let $H(w) = \int_a^w g(z) \cdot \mathrm{d}z$.

2.12: 2: This is a Newtonian system on the form

$$\dot{x} = y, \quad \dot{y} = -U'(x)$$

where $U(x) = x^2/2 - x^3/3$ and thus $H(x, y) = y^2/2 + U(x)$. The equilibrium points are for $y = 0$ and $U'(x) = 0$, thus $x = 0$ or $x = 1$. Since $x = 0$ is a local minimum for U and $x = 1$ is a local maximum, $(0, 0)$ becomes a center, and $(1, 0)$ a saddle.

2.12: 3: Here $U'(x) = x - x^3$, which yields $U(x) = x^2/2 - x^4/4$ (here and in the previous problem it is OK to throw away the constant of integration, which anyway does not influence the answer.) The equilibrium points are $x = 0$ and $x = \pm 1$, with a local minimum at $x = 0$ and local maxima at $x = \pm 1$. In other words, there is a center at $(0, 0)$ and saddles at $(\pm 1, 0)$.

2.12: 4: See the picture on the right, below.