75045 Dynamical systems 1998[1]

Final problem set: #7 — Solutions

3.2: 1: The system $\dot{x} = x - x^3$, $\dot{y} = -y$ is easy to analyze, because it consists of two independent ordinary differential equations. We see immediately that the equation for x has three equilibrium points: One unstable one at $x = 0$, and two stable ones at $x = \pm 1$. Similarly, the equation for y as one stable equilibrium point at $y = 0$. Thus, there is a saddle point at $(0, 0)$, and stable nodes at $(\pm 1, 0)$.

Every trajectory converges to one of the equilibria as $t \to \infty$, so in particular $\{(-1, 0), (0, 0), (1, 0)\}$ is an attracting set, and so is every invariant closed set containing these three points. In particular, the interval $[-1, 1]$ on the x axis is an attracting set, but not an attractor – since it does not contain a dense trajectory (it consists of the five trajectories $\{-1\}$, $(-1, 0)$, $\{0\}$, $(0, 1)$ and $\{1\}$, all on the x axis).[2]

The interval $(0, 1]$ on the x axis is no attractor because it is not closed. Of the infinite intervals mentioned in the problem, $(0, \infty)$ and $(-1, \infty)$ are disqualified from being called attracting sets because they are not closed. On the other hand, $[0, \infty)$, $[-1, \infty)$, and $(-\infty, \infty)$ are closed, and the latter two are attractors by the remarks at the top of the previous paragraph. $[0, \infty)$ is not attracting, for there are points arbitrarily close to 0 which are carried to $x = -1$ by the flow as $t \to \infty$.

3.2: 4:

[1] Because I use two-sided printing, this is page 19 even though there was no page 18. Think of page 18 as the blank backside of page 17.

[2] Here it becomes quite clear how confusing it can be using the same notation for points in the plane and for open intervals. At the beginning of the paragraph, $(-1, 0)$ is a point, while at the end $(-1, 0)$ is an interval!

3.3: 2: Polar coordinates yield

$$\dot r = \frac{x\dot x + y\dot y}{r} = r^2(1-r^2)^2,$$
$$\dot\theta = \frac{x\dot y - y\dot x}{r^2} = 1.$$

Since $\dot r > 0$ when $r \notin \{0,1\}$ the circle $r = 1$ will be stable on the inside and unstable on the outside.

3.4: 1: That $x = 2\cos 2t$, $y = \sin 2t$ solve the given system is trivially checked. (This would not be so hard to guess at: The common parenthesis in the two equations make it tempting to try setting $x = 2X$, $y = Y$, which yield a system for (X,Y) that is easily solved using polar coordinates. This would also decide the question of stability.)

We compute the divergence of this vector field:

$$\frac{\partial}{\partial x}\left[-4y + x\left(1-\frac{x^2}{4}-y^2\right)\right] + \frac{\partial}{\partial y}\left[x + y\left(1-\frac{x^2}{4}-y^2\right)\right] = -\frac{x^2}{2} - 2y^2 + 2\left(1-\frac{x^2}{4}-y^2\right)$$

with the value $-2 < 0$ when $x = 2\cos 2t$, $y = \sin 2t$. Hence this cycle is a stable limit cycle.

3.12: 7: We use Bendixson's formula $I_f(x_0) = 1 + (e-h)/2$ and find

Figure	e	h	$I_f(x_0)$
Left	3	3	1
Middle	0	6	-2
Right	2	0	2

3.12: 9: Clearly, since $z^k = 0$ or $\bar z^k = 0$ only for $z = 0$, the origin is the only equilibrium point for each of the systems $\dot z = z^k$, $\dot z = \bar z^k$. Whenever $r > 0$ the formula $z = re^{it}$ parametrizes a curve traveling once around the origin as t varies from 0 to 2π. This yields $z^k = r^k e^{ikt}$ and $\bar z^k = r^k e^{-ikt}$, which form an angle kt and $-kt$, respectively, with the x axis. This angle varies from from 0 to $2\pi k$ and $-2\pi k$, respectively, when t varies from 0 to 2π, so that the index of the circle becomes k and $-k$ respectively. This is therefore also the index of the critical point at 0.

Note that z^k is parallel with $z = re^{i\theta}$ precisely when $e^{ik\theta} = \pm e^{i\theta}$, which yields $e^{i(k-1)\theta} = \pm 1$. This holds when $(k-1)\theta = n$, i.e., for $\theta = n/(k-1)$, where we may put $n = 0, 1, \ldots, 2k-3$ (where I assume $k > 1$). The rays $z = re^{in/(k-1)}$ for $r > 0$ and $n = 0, 1, \ldots, 2k-3$ are therefore separatrices for $\dot z = z^k$; stable for even n, unstable for odd n. The sectors between these are all elliptic, and Bendixson's formula yields an index $1 + (e-h)/2 = 1 + (2k-2+0)/2 = k$ as expected.

Likewise, $\bar z^k$ is parallel with $z = re^{i\theta}$ precisely when $e^{-ik\theta} = \pm e^{I\theta}$, which yields $e^{i(k+1)\theta} = \pm 1$. This holds when $(k+1)\theta = n$, i.e., for $\theta = n/(k+1)$, where we may put $n = 0, 1, \ldots, 2k+1$. The rays $z = re^{in/(k+1)}$ for $r > 0$ and $n = 0, 1, \ldots, 2k+1$ are therefore separatrices for $\dot z = z^k$; unstable for even n, stable for odd n. The sectors between these are all hyperbolic, and Bendixson's formula yields an index $1 + (e-h)/2 = 1 + (0-(2k+2))/2 = -k$ as expected.