Exercise set B

Some exercises for TMA4230 Functional analysis

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Exercise B.1. Let K be a convex set. A point $x \in K$ is called an *extreme point* of K if there do not exist $y, z \in K$ with $y \neq z$ with $x = \alpha y + (1-\alpha)z$ where $0 < \alpha < 1$.

Show that the closed unit ball $\{u \colon \|u\| \le 1\}$ in c_0 has no extreme points, while the closed unit ball in c has some extreme points. Conclude that c_0 and c are not isometrically isomorphic, although their duals are. Also, show that c_0 and c are isomorphic. *Hint*: If $\|x\| \le 1$ and $|x_j| < 1$ for some j, show that x is not an extreme point of the closed unit ball of either c or c_0.

Exercise B.2. Let X be a metric space. A filter $\mathcal{F}$ on X is called a *Cauchy filter* if, for each $\varepsilon > 0$, there is $F \in \mathcal{F}$ so that $d(x, y) \le \varepsilon$ for each $x, y \in F$. (In other words, $\mathcal{F}$ contains sets of arbitrarily small diameter, where the diameter of F is $\sup\{d(x, y) \colon x, y \in F\}$.) Show that X is complete if and only if each Cauchy filter on X converges.

Let X be a uniformly convex Banach space, and let $f \in X^*$ with $\|f\| = 1$. Show that there exists a unique $x \in X$ with $\|x\| = 1$ and $f(x) = 1$. *Hint*: Show that the sets $\{x \in X \colon \operatorname{Re} f(x) > s\}$ where $s < 1$ generate a Cauchy filter.

Still assume X is uniformly convex, and $C \subseteq X$ a closed convex set. Show that C contains a unique point x_0 with smaller norm than any other member of C. *Hint*: Again, find a suitable Cauchy filter to solve the problem.

Exercise B.3. Let X be a normed space and $Z \subseteq X$ a closed subspace. All you really need to know about the *quotient space* X/Z is: It is another vector space, there is a linear map (which we write $x \mapsto [x]$) of X onto X/Z, and the null space of this map is Z.

Define a norm on X/Z by

$$\|[x]\| = \inf_{z \in Z} \|x + z\|.$$

Prove that this does in fact define a norm on X/Z. What goes wrong if Z is not closed? Prove that the mapping $x \mapsto [x]$ is an open map (without using the open mapping theorem – we have not yet assumed completeness).

Let $T \colon X \to Y$ be a bounded linear map with $T|_Z = 0$. Show that there exists a unique $\tilde{T} \colon X/Z \to Y$ with $\tilde{T}[x] = Tx$, and prove that $\|\tilde{T}\| = \|T\|$. (We call this *factoring* T *through* X/Z.)

Now, assume that X is complete. Prove that then X/Z is also complete. *Hint*: I find it easiest to prove that an absolutely convergent series is convergent.

Finally, pretend that you do not know a proof of the open mapping theorem and use the closed graph theorem to prove it. *Hint*: Use the closed graph theorem to prove the corollary to the open mapping theorem: that if a bounded linear map has an inverse then the inverse is bounded. In the general case, if $T \colon X \twoheadrightarrow Y$, factor T through X/Z where Z is the null space of T, and use previous results.