Solution set 2

to some problems given for TMA4230 Functional analysis

2004–02–11

Problem 4.5.5. The short version: $\big((ST)^*f\big)(x) = f\big(STx\big) = \big(S^*f\big)(Tx) = \big(T^*(S^*f)\big)(x) = \big((T^*S^*)f\big)(x)$. But perhaps it is more instructive to note that the definition of the adjoint can be written $S^*f = f \circ S$, where $\circ$ denotes the composition of functions. (When we write ST, that is really short for $S \circ T$.) So the identity we are asked to show is nothing but the obvious $f \circ (T \circ S) = (f \circ T) \circ S$.

Problem 4.5.8. In our notation, we are asked to prove $(T^*)^{-1} = (T^{-1})^*$. More precisely, assuming $T \in B(X,Y)$ is invertible, T^* is also invertible, with inverse $(T^{-1})^*$.

But equation (11) says $(ST)^* = S^*T^*$. Apply with $S = T^{-1}$ to get $(T^{-1})^*T^* = I^* = I$. And apply that with T and T^{-1} interchanged, to get $T^*(T^{-1})^* = I$. Together, these two show that T^* and $(T^{-1})^*$ are each other's inverses.

Problem 4.5.9. Note that any set and its closure have the same annihilator: Just recall that the null space of any bounded linear functional is closed. So what we are asked to prove is just[1]

$$\mathcal{R}(T)^\perp = \mathcal{N}(T^*).$$

Now, if $f \in Y^*$ then[2]

$$\begin{aligned} f \in \mathcal{R}(T)^\perp &\Leftrightarrow f(Tx) = 0 \qquad \forall x \in X \\ &\Leftrightarrow (T^*f)(x) = 0 \qquad \forall x \in X \\ &\Leftrightarrow T^*f = 0 \\ &\Leftrightarrow f \in \mathcal{N}(T). \end{aligned}$$

Problem 4.5.10. Take a typical element Tx of $\mathcal{R}(T)$, where $x \in X$. We need to show that[3] $Tx \in \mathcal{N}(T^*)_\perp$. Thus, we take $f \in \mathcal{N}(T^*)$, and must prove that $f(Tx) = 0$. But then $f(Tx) = (T^*f)(x) = 0$ since $T^*f = 0$.

Problem 4.6.3. Assume that X is reflexive. This means that every linear functional on X^* is of the form $\hat{x}(f) = f(x)$, where $x \in X$. Moreover, the canonical map $C\colon x \mapsto \hat{x}$ is an isometric isomorphism of X onto X^{**}. Briefly, X^{**} "is" X. So its dual X^{***} "is" X^*, which means X^* is reflexive.

To make this precise, we must, like Bill Clinton, ask what the meaning of "is" is.

Consider $\varphi \in X^{***}$: a bounded linear functional on X^{**}. Then $x \mapsto \varphi(\hat{x})$ is a bounded linear functional on X. So this defines $f \in X^*$ by the formula

$$f(x) = \varphi(\hat{x}).$$

But recalling the definition of $\hat{x}$, this is

$$\varphi(\hat{x}) = \hat{x}(f), \qquad x \in X.$$

Since *every* member of X^{**} is of the form $\hat{x}$, this means

$$\varphi(\xi) = \xi(f), \qquad \xi \in X^{**}$$

and that every $\varphi \in X^{***}$ can be written in this way, for some $f \in X^*$, is precisely the reflexivity of X^*.

Problem 4.6.8 and 9. (I have in fact covered this in the lectures, but I will post a solution anyway. Later.)

[1] Recall that I write $M^\perp$ for the annihilator, where Kreyszig writes M^a.

[2] $\forall$ is short for "for all".

[3] What Kreyszig calls the annihilator ${}^aB$, I prefer to call the *preannihilator* and write as $B_\perp$ (the annihilator of a subset of X^* is contained in X^{**}).