Solution set 3

to some problems given for TMA4230 Functional analysis

2004–02–20

Problem 4.6.8. The set of $f \in X^*$ such that $f|_M = 0$ is what I have called the annihilator $M^\perp$ of M. And the set of $x \in X$ so that $f(x) = 0$ for each $f \in M^\perp$ is the preannihilator $(M^\perp)_\perp$ of $f \in M^\perp$. In this notation, we are asked to prove that $\overline{\operatorname{span} M} = (M^\perp)_\perp$.

First, that $M \subseteq (M^\perp)_\perp$ is trivial: It just says that if $x_0 \in M$ and if $f(x) = 0$ for every $x \in M$, then $f(x_0) = 0$. But $(M^\perp)_\perp$ is a subspace of X, so then $\operatorname{span} M \subseteq (M^\perp)_\perp$ as well. Finally, $(M^\perp)_\perp$ is closed, so $\overline{\operatorname{span} M} \subseteq (M^\perp)_\perp$.

Conversely, assume that $x_0 \notin \overline{\operatorname{span} M}$. By the Hahn–Banach theorem (or rather a consequence of it – see Lemma 4.6-7) there is a functional $f \in X^*$ with $f(x_0) \neq 0$ and $f|_M = 0$. Thus $f \in M^\perp$, and then $f(x_0) \neq 0$ implies $x_0 \notin (M^\perp)_\perp$.

Problem 4.6.9. Recall (Kreyszig p. 168) that M being *total* in X means $\overline{\operatorname{span} M} = X$. In the notation introduced above, we are asked to show that M is total if and only if $M^\perp = \{0\}$.

If M is total then $M^\perp = \{0\}$ is an obvious consequence: For any bounded linear functional f which vanishes on M vanishes on $\operatorname{span} M$ (because f is linear), and then it vanishes on $\overline{\operatorname{span} M}$ (because f is continuous).

Conversely, if $M^\perp = \{0\}$ then by the previous problem $\overline{\operatorname{span} M} = (M^\perp)_\perp = \{0\}_\perp = X$, so M is total.

Problem 4.7.5. More generally, a subset of X is dense if and only if its complement has empty interior. (The statement of the problem follows from this just by using the definition of *rare*.)

Let $A \subset X$. Then A is dense in X if and only if $A \cap U \neq \emptyset$ for every nonempty open set $U \subseteq X$. But $A \cap U \neq \emptyset$ is the same as saying U is not contained in the complement of A. And saying that a set contains no open set is the same as saying it has empty interior.

Problem 4.7.6. If the complement M^{c} of a meager set M is meager, then we have written X as a union of two meager sets M and M^{c}. By definition each is a countable union of rare sets. Joining two countable sets of rare sets we again get a countable set of rare sets, which cannot have union X by the Baire theorem. This is a contradiction.

Problem 4.7.7. This problem just states the contrapositive[1] of the uniform boundedness theorem. So there really is nothing to do here. (But it is useful to have the theorem in this form.)

Problem 4.7.8. Using the notation (almost)[2] introduced in the problem, if $x \in X$ with $x_j = 0$ when $j \geq J$, then $f_n(x) = 0$ if $n > J$, otherwise $|f_n(x)| = n\,|x_j| \leq J\|x\|_\infty$. So the family $(f_n)_{n=1}^\infty$ is pointwise bounded. However, it is not uniformly bounded, for $\|f_n\| = n$.

Extra: *Prove that a closed subspace of a reflexive space is reflexive.*

Let X be a reflexive space and $Y \subseteq X$ a closed subspace. Assume $\eta \in Y^{**}$. Define $\xi \in X^{**}$ by setting

$$\xi(f) = \eta(f|_Y) \qquad (f \in X^*).$$

Since X is reflexive, the functional ξ is really of the form $f \mapsto f(x)$ for some $x \in X$. So the above definition becomes

$$\eta(f|_Y) = f(x) \qquad (f \in X^*).$$

We claim that $x \in Y$. For if $x \notin Y$, there is a bounded linear functional on X with $f|Y = 0$ and $f(x) \neq 0$ (because Y is closed, see Lemma 4.6-7). But this is impossible since then $0 \neq f(x) = \eta(f|_Y) = \eta(0) = 0$.

So we now write

$$\eta(g) = g(x) \qquad (g = f|_Y,\ f \in X^*).$$

But, by the Hahn–Banach theorem, *every* bounded linear functional on Y can be written $f|_Y$ with $f \in X^*$. Thus $\eta(g) = g(x)$ for all $g \in Y^*$, where $x \in Y$. This proves that Y is reflexive.

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[1] The *contrapositive* of a statement on the form “if A then B” is the equivalent statement “if not B then not A”.

[2] I dislike the convention of using different letters for a vector and its components, as in $x = (\xi_j)$. There aren't enough letters in the alphabet, and this is wasteful.