Solution set 6

to some problems given for TMA4230 Functional analysis

2004–03–15

Problem 4.3.7. (See the back of the book for solutions to this and other odd-numbered exercises.)

Problem 4.3.14. (The space X must be a *real* space for this to make sense, for a hyperplane in a complex space does not divide the space in two pieces – the point being that 0 divides $\mathbb{R}$ into the two half lines, but 0 does not divide $\mathbb{C}$; i.e., $\mathbb{C} \setminus \{0\}$ is connected.)

Writing the desired hyperplane H_0 as $\{x \in X : f(x) = r\}$, we are looking for $f \in X^*$ with $f(x) \le r$ when $x \in S_r(0)$ while $f(x_0) = r$. This is fulfilled by the functional $\tilde{f}$ of Theorem 4.3-3.

Problem 4.3.15. Once more, see the back of the book for solutions to this and other odd-numbered exercises. But also note that this result follows from the isometry of the canonical map from X into X^{**} (Lemma 4.6-1).

Problem 4.7.10. Let f_n be the linear functional on c_0 given by $f_n(x) = \sum_{j_1}^n x_j y_j$.[1] The condition of the problem means that $\lim_{n\to\infty} f(x)$ exists for each $x \in c_0$. In particular, since a convergent sequence is bounded, there is for each $x \in c_0$ a constant c_x so that $|f_n(x)| < c_x$ for each n. By the Banach–Steinhaus theorem (also known as the Uniform boundedness theorem, Theorem 4.7-3) there is a uniform bound $\|f_n\| \le c$ for all the functionals f_n. Since $\|f_n\| = \sum_{j=1}^n |y_j|$, this implies $\sum_{j=1}^\infty |y_j| \le c < \infty$.

Problem 4.7.13. (See the back of the book for solutions to this and other odd-numbered exercises.)

Problem 4.7.14. Clearly, $(a) \Rightarrow (b)$ because $\|T_n x\| \le \|T_n\|\|x\|$, and $(b) \Rightarrow (c)$ because $|g(T_n x)| \le \|g\|\|T_n x\|$.

Next, $(c) \Rightarrow (b)$ by Problem 4.7.13, with $T_n x$ replacing x_n and g replacing f.

And finally, $(b) \Rightarrow (a)$ by the Banach–Steinhaus theorem (Theorem 4.7-3).

Problem 4.12.10. Consider the identity map on X, but think of it as a linear map $\iota\colon X_1 \to X_2$. Then the inclusion $\mathcal{T}_1 \supseteq \mathcal{T}_2$ implies that ι is continuous, and hence bounded. By the open mapping theorem, ι^{-1} is also bounded, and hence continuous. This implies $\mathcal{T}_1 \subseteq \mathcal{T}_2$, and hence $\mathcal{T}_1 = \mathcal{T}_2$.

Exercise B.1. Assume $\|x\| \le 1$ and $|x_j| < 1$ for some j. (Of course, the norm here is really $\|\cdot\|_\infty$.) We can write the component x_j as a convex combination $tw - (1-t)w$ of w and $-w$, where $|w| = 1$ and $t = \frac{1}{2}\bigl(1 + |x_j|\bigr)$, so $0 < t < 1$. (If $x_j = 0$, just put $w = 1$ and $t = \frac{1}{2}$ instead.) Let y and z have all components equal to those of x except the j-th component, where $y_j = w$ and $z_j = -w$. Then $\|y\| = \|z\| = 1$, and $x = ty + (1-t)z$. Also, y and z are different from x, so x is not an extreme point of the unit ball.

Every vector x in c_0 has a component with $|x_j| < 1$, since $x_j \to 0$ as $j \to \infty$. Thus c_0 has no extreme points.

But c has many extreme points: In fact every member $x \in c$ with $\|x\| = 1$ and the property that $|x_j| = 1$ for every j is an extreme point of the unit ball. The reason is that $[-1, 1]$ (for real scalars) or $\{w \in \mathbb{C} : |w| \le 1\}$ (for complex scalars) have as *their* extreme boundaries precisely the scalars with $|w| = 1$. Thus, if $x = ty + (1-t)z$ with $\|y\| \le 1$ and $\|z\| \le 1$ and $0 < t < 1$, then also $x_j = ty_j + (1-t)z_j$ with $|y_j| \le 1$ and $|z_j| \le 1$, and this implies $x_j = y_j = z_j$. Thus $x = y = z$.

An isometric isomorphism maps one unit ball bijectively onto the other, and therefore it also maps extreme points to extreme points. Thus c and c_0 cannot be isometrically isomorphic. (But they both have ℓ^1 for their dual.)

Exercise B.2. First, if every Cauchy filter converges then so does any Cauchy sequence: For if (x_j) is a Cauchy sequence then, by definition, the diameter of a tail $\{x_j : j \ge n\}$ can be made as small as we wish merely by chosing n large enough. So the filter generated by the tails of the sequence is Cauchy, and hence convergent. But the convergence of this filter is the same as convergence of the sequence.

Conversely, assume that X is complete, and let $\mathcal{F}$ be a Cauchy filter on X. For each n, let $A_n \in \mathcal{F}$ have diameter less than $1/n$. Let $F_n = A_1 \cap \ldots \cap A_n \in \mathcal{F}$. Thus $F_1 \supseteq F_2 \supseteq \cdots$, and $\operatorname{diam} F_n < 1/n$. Pick $x_n \in F_n$ for each n. Since the tail $\{x_n, x_{n+1}, \ldots\} \subseteq F_n$, the diameter of this tail is also less than $1/n$. Thus (x_j) is a Cauchy sequence, and hence convergent. Let x be its limit. If $\varepsilon > 0$ and n is big enough, $\|x - x_n\| < \varepsilon$. Make

[1] As usual, I ignore Kreyszig's preference for using different letters for a vector and its components, as in $x = (\xi_j)$, writing $x = (x_j)$ instead.

n even bigger, if necessary, so that also $1/n < \varepsilon$. Then, if $y \in F_n$, $\|x-y\| \le \|x-x_n\| + \|x_n - y\| < 2\varepsilon$. Thus $\mathcal{F} \to x$.

Now let X be a uniformly convex Banach space, $f \in X^*$, and $\|f\| = 1$. By definition, for each $s < 1$ there is at least one $x \in X$ with $\|x\| = 1$ and $f(x) > s$. Thus, the set $F_s = \{x \in X \colon \|x\| = 1 \text{ and } f(x) > s\}$ is nonempty.[2] Since the collection of sets F_s is totally ordered by inclusion, it generates a filter $\mathcal{F}$. Then $\mathcal{F}$ is in fact a Cauchy filter: For if $\varepsilon > 0$ there is some $\delta > 0$ so that whenever $\|x\| = \|y\| = 1$ and $\|x+y\| > 2-\delta$, then $\|x-y\| < \varepsilon$. Let $s = 1 - \frac{1}{2}\delta$. If $x, y \in F_s$ then $f(x+y) > 2s = 2-\delta$. Since $\|f\| = 1$ this implies $\|x+y\| > 2-\delta$, and so $\|x-y\| < \varepsilon$. Thus $\operatorname{diam} F_s \le \varepsilon$, and so $\mathcal{F}$ is Cauchy. Let x be the limit of $\mathcal{F}$. Since the norm as well as f are continuous, we find $\|x\| = f(x) = 1$. This is unique, for if $\|y\| = f(y) = 1$ as well, then as in the above argument we find $\|x+y\| = 2 > 2-\delta$ for *any* $\delta > 0$, so that $\|x-y\| < \varepsilon$ for any $\varepsilon > 0$.

Finally, let X be uniformly convex and $C \subseteq X$ closed and convex. Let $d = \inf\{\|x\| \colon x \in C\}$, and let $F_s = \{x \in C \colon \|x\| < s\}$. Then $F_s \ne \emptyset$ for each $s > d$, and so $\{F_s \colon s > d\}$ generates a filter $\mathcal{F}$ as before. We need to normalize elements of F_s in order to use the uniform convexity.

So let now $x, y \in F_s$ and write $\tilde{x} = x/\|x\|$, $\tilde{y} = y/\|y\|$, so that $\|\tilde{x}\| = \|\tilde{y}\| = 1$. Furthermore,

$$\tilde{x} + \tilde{y} = \frac{x}{\|x\|} + \frac{y}{\|y\|} = \alpha\big(tx + (1-t)y\big), \qquad \alpha = \frac{1}{\|x\|} + \frac{1}{\|y\|}, \quad t = \frac{1}{\alpha\|x\|}, \quad 1-t = \frac{1}{\alpha\|y\|}.$$

Thus, since C is convex, we have $tx + (1-t)y \in C$, so that $\|tx + (1-t)y\| \ge d$, and therefore $\|\tilde{x} + \tilde{y}\| \ge \alpha d$. But by assumption $\|x\| < s$ and $\|y\| < s$, so $\alpha > 2/s$, and therefore $\|\tilde{x} + \tilde{y}\| > 2d/s$.

Let $\varepsilon > 0$, and pick $\delta > 0$ as in the definition of uniform convexity. If $s > d$ is small enough then $\|\tilde{x} + \tilde{y}\| > 2d/s \ge 2-\delta$, so that $\|\tilde{x} - \tilde{y}\| < \varepsilon$. Then (subtracting and adding $\|x\|\tilde{y}$ in the second step)

$$\|x-y\| = \big\|\|x\|\tilde{x} - \|y\|\tilde{y}\big\| \le \big\|\|x\|(\tilde{x} - \tilde{y})\big\| + \big\|(\|x\| - \|y\|)\tilde{y}\big\| = \|x\|\|\tilde{x} - \tilde{y}\| + \big|\|x\| - \|y\|\big| < s\varepsilon + s - d,$$

which can be made as small as we wish by making ε and $s-d$ small enough (where ε must be chosed first, and s next).

This proves that $\mathcal{F}$ is a Cauchy filter, and its limit will be the point in C with smallest norm. (We skip the finish of the proof, which is very similar to that for the previous question.)

Exercise B.3. To prove that the proposed norm on the quotient space is well defined, we note that, when $[x] = [y]$, that means $[x-y] = 0$ (by linearity of the map $x \mapsto [x]$), so $x - y \in Z$, and the sets $\{x + z \colon z \in Z\}$ and $\{y + z \colon z \in Z\}$ used to define $\|[x]\|$ and $\|[y]\|$, respectively, are the same.

Clearly, $\|0\| = 0$, and if $[x] \ne 0$ then $\|[x]\| = \inf_{z\in Z}\|x - z\| = \operatorname{dist}(x, Z) > 0$ since Z is closed. If x and y are arbitrary and $z, w \in Z$ then $\|[x+y]\| \le \|x+y+z+w\| \le \|x+z\| + \|y+w\|$. Taking infimums over z and w we obtain the triangle inequality $\|[x] + [y]\| \le \|[x]\| + \|[y]\|$. The equality $\|\alpha[x]\| = |\alpha|\|[x]\|$ is trivially true when $\alpha = 0$. When $\alpha \ne 0$ it follows from the fact that $\{\alpha z \colon z \in Z\} = Z$.

If Z is not closed then there is some $x \in \overline{Z}$ but not in Z. Then $\|[x]\| = 0$ but $[x] \ne 0$, which violates one of the axioms for the norm – in fact, we get a *seminorm* but not a norm.

If $\|[x]\| < 1$ then $\|x + z\| < 1$ for some $z \in Z$, by the definition of $\|[x]\|$. Then $[x] = [x+z]$. Thus the open unit ball in X/Z is contained in the image of the open unit ball in X, which shows that the quotient map $X \to X/Z$ is open. To this we might have added the obvious fact that $\|[x]\| \le \|x\|$, so the quotient map has norm equal to 1.

Since the equation $\tilde{T}[x] = Tx$ prescribes the value of $\tilde{T}$ on every element of X/Z, this of course defines $\tilde{T}$ uniquely, if it defines $\tilde{T}$ at all. To show that it does, assume $[x] = [y]$. Then $x - y \in Z$, so $T(x-y) = 0$. Thus $Tx = Ty$.

Clearly $\|Tx\| = \|\tilde{T}[x]\| \le \|\tilde{T}\|\|[x]\| \le \|\tilde{T}\|\|x\|$, so $\|T\| \le \|\tilde{T}\|$. Moreover, for any $z \in Z$ then $\|\tilde{T}[x]\| = \|\tilde{T}[x+z]\| = \|T\|\|x+z\|$. Taking the infimum over all $z \in Z$, we get $\|\tilde{T}[x]\| \le \|T\|\|[x]\|$, so that $\|\tilde{T}\| \le \|T\|$.

Let $w_j \in X/Z$ with $\sum\|w_j\| < \infty$. Pick $x_j \in X$ with $w_j = [x_j]$ and $\|x_j\| < \|w_j\| + 2^{-j}$. Then $\sum\|x_j\| < 1 + \sum\|w_j\| < \infty$. Since X is complete, $\sum x_j$ converges. Hence so does $\sum w_j$, since $\sum_{j=1}^n w_j = \sum_{j=1}^n [x_j] = \big[\sum_{j=1}^n x_j\big]$, the partial sums on the righthand side converges, and the quotient map is continous.

Now let $T \colon X \to Y$ be a bounded linear map, where X and Y are complete. Assume also that T has an inverse $T^{-1} \colon Y \to X$. The graph of T^{-1} is the image of the graph of T under the map $X \times Y \to Y \times X$ which interchanges the components. But this map is an isomorphism, and so it maps closed subsets to closed subsets. Thus the graph of T^{-1} is closed, and the closed graph theorem implies that T^{-1} is bounded.

Finally, let $T \colon X \twoheadrightarrow Y$ be a bounded linear map *onto* Y. Let Z be the kernel (null space) of T. Let $Q \colon X \twoheadrightarrow X/Z$ be the quotient map, and let $\tilde{T} \colon X/Z \twoheadrightarrow Y$ be the factored map defined above, so that $T = \tilde{T}Q$. Now $\tilde{T}$ has an inverse, which must be bounded by the above paragraph. In particular it is open. We have also shown that Q is open. Thus $T = \tilde{T}Q$ is open.

[2] I forgot the condition $\|x\| = 1$ in the hint, but I hope you could figure out to add this condition yourself. Also, in the case of complex scalars, every occurence of f here should actually be replaced by $\operatorname{Re} f$.