Norwegian University of Science and Technology
Department of Mathematical Sciences

For questions during the exam:
Harald Hanche-Olsen tel. 73 59 35 25

Exam in TMA4230 Functional analysis

Tuesday 31 May 2005
09:00 – 13:00

Permitted aids (code D): Simple calculator (HP 30S)

Note: A Norwegian text is appended.
Answers in English, Norwegian, or a mixture of the two accepted.

Grades available: 21 June 2005

Problem 1

a) State (but do not prove) the closed graph theorem, taking care to get its conditions right.

A linear operator P on a vector space X is called a *projection* if $P^2 = P$.

Two subspaces Y, Z of X are called *complementary* if $Y \cap Z = \{0\}$ and $X = Y + Z$ (that is, if every $x \in X$ can be written as $x = y + z$ for unique $y \in Y$, $z \in Z$).

b) Let P be a bounded projection on a normed space X. Show that the image (range) $\operatorname{im} P$ and kernel (null space) $\ker P$ are closed, complementary subspaces of X.

c) Conversely, let Y and Z be closed, complementary subspaces of a Banach space X. Show that there is a bounded projection on X whose image is Y and whose kernel is Z. *Hint*: You must write $P(y + z) = y$ for $y \in Y$ and $z \in Z$. Use the closed graph theorem.

Problem 2

a) State (but do not prove) the spectral mapping theorem for polynomials applied to members of an algebra with unit.

b) Assume that a member x of an algebra with a unit e satisfies a polynomial identity of the form

$$x^n + a_{n-1}x^{n-1} + \cdots + a_1 x + a_0 e = 0.$$

Show that then the spectrum of x is finite. As a special case, show that if $x^4 - 1 = 0$ then $\sigma(x) \subseteq \{-1, +1, -i, +i\}$.

Problem 3 Assume $1 < p < \infty$, and let U and V be disjoint open and convex subsets of $L^p(\Omega, \mu)$. Then there exists some $w \in L^q(\Omega, \mu)$ (where p and q are conjugate exponents) so that

$$\int_\Omega uw \, d\mu < \int_\Omega vw \, d\mu$$

for all $u \in U$ and $v \in V$.

Prove this. Explain what general theorems you are using, and write up the statement of those theorems.

Problem 4

a) State (but do not prove) the spectral theorem for self-adjoint bounded operators on a Hilbert space. Also state the defining properties of a spectral family, and briefly state the definition of the integral $\int \lambda \, dE_\lambda$ occurring in the spectral theorem. Finally, write the corresponding integral for $f(T)$ where $f \in C\bigl(\sigma(T)\bigr)$.

b) Assume that the self-adjoint bounded operator T has finite spectrum: Say $\sigma(T) = \{\lambda_1, \ldots, \lambda_n\}$ where $\lambda_1 < \lambda_2 < \cdots < \lambda_n$. What will the corresponding spectral family look like? Rewrite the integral in the spectral theorem as a sum.