Solution set 3

to some problems given for TMA4230 Functional analysis

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Note: In my solutions the two "warmup" exercises from Kreyszig, I have replaced the subspace Y by N for consistency with the remaining problems.

Problem 2.1.14. Cosets form a partition of X: This means that every element of X is a member of some coset (in fact $x \in x + N$ since $y \in N$), and distinct cosets are disjoint (in fact, if $x \in (u + N) \cap (v + N)$, then $x - u \in N$ and $x - v \in N$ so that $u - v = (x - v) - (u - v) \in N$, and $u + N = v + N$ follows).

Checking the vector space axioms for X/N is easy and I will not do it here. A more important point is to check that the given vector space operations are *well defined*: That is, that the sum $(w + N) + (x + N) = (w + x) + N$ as defined in the problem does not depend on the particular choice of w and x used to represent their respective cosets. This is not hard either, but it *is* important.

Problem 2.3.14. An equivalent way to write the definition of the quotient norm is[1]

$$\|[x]\| = \inf_{w\in[x]} \|w\|$$

where $[x]$ is just shorthand notation for the coset $x + N$. Note that $w \in [x] \Leftrightarrow w - x \in N$. In fact, if we write $w = x - y$ with $y \in N$, the definition becomes

$$\|[x]\| = \inf_{y\in N} \|x - y\|,$$

which is just the distance from x to N.

In particular, if $[x] \neq 0$ then $x \notin N$, so that distance is positive (since N is closed), and so $\|[x]\| > 0$.

For a scalar $c \neq 0$ we get

$$\|c[x]\| = \|[cx]\| = \inf_{y\in N} \|cx - y\| = \inf_{y\in N} \|cx - cy\| = |c| \inf_{y\in N} \|x - y\| = |c|\|[x]\|$$

where we have used $cy \in N \Leftrightarrow y \in N$. The equality holds for $c = 0$ as well, though the above calculation makes less sense then.

Finally, for the triangle inequality, note that whenever $u' \in [u]$ and $v' \in [v]$ then $u' + v' \in [u+v]$, so that $\|[u+v]\| \le \|u'\|+\|v'\|$. Take the infimum over all $u' \in [u]$ and $v' \in [v]$ to conclude $\|[u + v]\| \le \|[u]\| + \|[v]\|$.

Problem. *Assume that X is a normed space and $N \subseteq X$ is a closed subspace. Show that the* canonical map $Q\colon X \to X/N$ (defined by $Q(x) = [x] = x + N$) is open.

Solution. If $\|Q(x)\| < 1$ then by construction of the norm, there exists some $w \in X$ with $\|w\| < 1$ and $Q(x) = Q(w)$. Thus Q maps the open unit ball of X onto the open unit ball of X/N, and so Q is open.

Problem. *Assume furthermore that $T\colon X \to Y$ is bounded, and $N \subseteq \ker T$. Show that there is a unique linear map $R\colon X/N \to Y$ so that $T = RQ$. What is its norm?*

Solution. The requirement $T = RQ$ becomes $Tx = R[x]$ for every $x \in X$. Since every member of X/N is of the form $[x]$, this shows the uniqueness of R (if it exists).

We must show that $R[x] = Tx$ is *well defined.* If $[x] = [w]$ then $x - w \in N$. Then by assumption $T(x - w) = 0$, so $Tx = Tw$. This proves that R is well defined.

It remains to prove that R is linear: But $R([w] + [x]) = R[w + x] = T(w + x) = Tw + Tx = R[w] + R[x]$, and $R(c[x]) = R[cx] = T(cx) = cTx = cR[x]$.

[1] I am dropping Kreyszig's subscript 0 on the quotient norm.

Problem. *Assume furthermore* (still!) that $N = \ker T$. Show that T is open if and only if R has a bounded inverse.

Solution. If R has a bounded inverse then R is open. Since we have already proved that Q is open, it follows that $T = RQ$ is open.

On the other hand, if T is open then there exists M so that any $y \in Y_1$ (the closed unit ball of Y) can be written $y = Tx$ with $x \in X$ and $\|x\| < M$. But then $y = RQx$, and $\|Qx\| \le \|x\| \le M$. Thus R is open. In particular R maps X/N onto Y. R is also injective, since $R[x] = 0 \Leftrightarrow Tx = 0 \Leftrightarrow x \in N \iff [x] = 0$. An bijective open map has a bounded inverse.

Problem. *Finally, a challenge: Use the closed graph theorem to prove the open mapping theorem.* (Hint: *Do it first for one-to-one mappings, then use the above results to get the general case.*)

Solution. Assume X and Y are Banach spaces and $T\colon X \to Y$ is bounded, onty Y, and one-to-one. Thus T has an inverse, and the graph of the inverse is $\{(Tx, x)\colon x \in X\}$, which is closed. (It is the image of the graph $\{(x, Tx)\colon x \in X\}$ of T under the isometry $X \times Y \to Y \times X$ given by $(x, y) \mapsto (y, x)$.) By the closed graph theorem then, T^{-1} is bounded, and so T is open.

For the general case, let $T\colon X \to Y$ be bounded and onto Y. Write $T = RQ$ as in the previous problems, where $N = \ker T$.

Now R is bounded, one-to-one and onto, so it has a bounded inverse by the first part. Thus T is open by the previous problem.