# Solution set 5

to some problems given for TMA4230 Functional analysis

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**Exercise B.1.** I will not write up the proof that $d_p$ is a metric. I gave so many hints, the rest is just elementary calculus.

Now, let $f$ be a linear functional on $L^p$, where $0 < p < 1$. If $f$ is continuous then (picking $\varepsilon = 1$) there is some $\delta > 0$ so that $d_p(x, 0) < \delta \Rightarrow |f(x)| < 1$. In other words, $\|x\|_p < \delta^{1/p} \Rightarrow |f(x)| < 1$. Thus $\|x\|_p < 1 \Rightarrow |f(x)| < \delta^{-1/p}$, so $f$ is bounded. Conversely, if $f$ is bounded, say $\|x\|_p < 1 \Rightarrow |f(x)| < M$, then in a similar way we find $d_p(x, 0) < (\varepsilon/M)^p \Rightarrow |f(x)| < \varepsilon$, so $f$ is continuous at 0. Since the metric $d_p$ is translation invariant ($d_p(x + z, y + z) = d(x, y)$), it follows that $f$ is continuous everywhere.

Now let $f$ be a nonzero continuous linear functional on $L^p$. As remarked in the problem, we may replace $f$ by a multiple of itself and so assume

$$\sup_{\|u\|_p=1} |f(u)| = 1. \tag{1}$$

(At the outset, the supremum is finite because $f$ is bounded, and it is positive because $f$ is nonzero.)

Let $u \in L^p$ with $\|u\|_p = 1$. If $u = u_1 + u_2$ and $u_1 u_2 = 0$, that means there is a measurable subset $E$ of $[0, 1]$ so that $u_2$ is zero on $E$ and $u_1$ is zero on $E^c = [0, 1] \setminus E$. Thus

$$\begin{aligned} \|u\|_p^p &= \int_0^1 |u|^p\,dx = \int_E |u|^p\,dx + \int_{E^c} |u|^p\,dx = \int_E |u_1|^p\,dx + \int_{E^c} |u_2|^p\,dx \\ &= \int_0^1 |u_1|^p\,dx + \int_0^1 |u_2|^p\,dx = \|u_1\|_p^p + \|u_2\|_p^p. \end{aligned}$$

In order to get $\|u_1\|_p^p = \|u_2\|_p^p = \frac{1}{2}$ all we need is to determine $E$ so that $\int_E |u|^p\,dx = \frac{1}{2}$. But just pick $E = [0, t]$, notice that then the integral is a continuous function of $t$ which increases from 0 to 1, and use the intermediate value theorem.

Thus, for $k = 1, 2$ we find $\|2^{1/p} u_k\|_p^p = 1$ so that $|2^{1/p} f(u_k)| \le 1$. Thus $|f(u)| \le |f(u_1)| + |f(u_2)| \le 2^{-1/p} + 2^{-1/p} = 2^{1-1/p}$. Since $0 < p < 1$ then $1 - 1/p < 0$, so $2^{1-1/p} < 1$. But then this contradicts (1).

**Exercise B.2.** First, let $X$ be an ordered vector space with $X^+ = \{x \in X : x \ge 0\}$.

Then $0 \in X^+$ because $0 \ge 0$. Given a scalar $c \ge 0$ and vector $x \in X^+$, we find $cx \in X^+$ because $c \ge 0$ and $x \ge 0$ imply $cx \ge c0 = 0$.

If $x \in X^+ \cap (-X^+)$ then $x \ge 0$ and $-x \ge 0$. Adding $x$ to the latter inequality we get $0 \ge x$, and $x = 0$ follows.

If $x, y \in X^+$ then we can add $y$ to $x \ge 0$ to get $x + y \ge y$. Since also $y \ge 0$ we get $x + y \ge 0$, so $x + y \in X^+$.

To show that $X^+$ is convex, apply the previous result to $tx$ and $(1 - t)y$, where $t \in [0, 1]$.

And finally, assuming convexity and $x, y \in X^+$, we find $\frac{1}{2}(x + y) \in X^+$ by convexity. Multiply by 2 to conclude $x + y \in X^+$.

Now let $X^+$ be a proper convex cone in $X$, and define $x \le y \Leftrightarrow y - x \in X^+$. Then $x \le x$ becuse $0 \in X^+$, $x \le y$ and $y \le x$ imply $x = y$ because $x - y \in X^+ \cap (-X^+) = \{0\}$, and $x \le y \le z$ implies $x \le z$ because $z - x = (z - y) + (y - x)$ with $z - y \in X^+$ and $y - x \in X^+$. So far, we have shown that $\le$ is a partial order. If $x \le y$ and $c \ge 0$ then $cy - cx = c(y - x) \in X^+$, so $cx \le cy$. Also $(y + v) - (x + v) = y - x \in X^+$, so $x + v \le y + v$.

**Exercise B.3.** First, if $-ce \le x \le ce$ then $c \ge 0$, since $e \in X^+$. Thus $\|x\| \ge 0$.

If $\|x\| = 0$ then $x \le ce$ for all $c > 0$. Thus $c^{-1}x \le e$ for all $c > 0$, and so $x \le 0$ by the second order unit axiom. Similarly, $x \ge -ce$ for all $c > 0$, or $-cx \le e$ for all $c > 0$, which implies $-x \le 0$, so $x \ge 0$. We conclude $x = 0$ when $\|x\| = 0$.

When $-ce \le x \le ce$ and $-de \le y \le de$ we find $-(c + d)e \le x + y \le (c + d)e$. Taking the infimum over all $c$ and $d$ we get $\|x + y\| \le \|x\| + \|y\|$.

Multiplying the inequality $-ce \le x \le ce$ by a real number $t \ne 0$ we find the equivalent $-cte \le tx \le cte$ (even if $t < 0$). Thus $\|tx\| = |t|\|x\|$ follows.

We have shown that $\|\cdot\|$ is a norm.

Clearly, we have $x \le ce$ for every $c > \|x\|$. Write this inequality as $x - \|x\|e \le (c - \|x\|)e$, so that $x - \|x\|e \le te$ for all $t > 0$. Thus $x - \|x\|e \le 0$ by the second order unit axiom. In other words, $x \le \|x\|e$. The inequality $x \ge -\|x\|e$ follows in a similar way, or more simply by replacing $x$ by $-x$.

**Exercise B.4.** From the definition of the state space we immediately get $x \le y \Rightarrow f(x) \le f(y)$, when $f \in S$. Thus $-\|x\|f(e) \le f(x) \le \|x\|f(e)$ follows, and therefore $|f(x)| \le \|x\|$ since $f(e) = 1$. Thus $\|f\| \le 1$. But also $\|f\| \ge 1$ because $f(e) = 1 = \|e\|$. Thus $S$ is a subset of the closed unit ball of $X^*$. It is a weakly* closed subset, because of the way it is defined in terms of weakly* continuous functionals $f \mapsto f(x)$ with $x \in X$. Since the closed unit ball of $X^*$ is weakly* compact, then so is the weakly* closed subset $S$.

**Exercise B.5.** Certainly, if $x \in X$ and $x \ge 0$ then $f(x) \ge 0$ for all $f \in S$, by the very definition of $S$.

Assume now instead $x \not\ge 0$, but still $f(x) \ge 0$ for all $f \in S$. We shall use the Hahn–Banach separation theorem to separate $x$ from $X^+$. Actually, we need a little bit more: We really should separate a neighbourhood of $x$ from $X^+$. Certainly, we can find some $\varepsilon > 0$ so that $x + \varepsilon e \notin X^+$. For otherwise $-x \le \varepsilon e$ for every $\varepsilon > 0$, which would imply $-x \le 0$, i.e., $x \ge 0$. So now the $\varepsilon$-ball $B_\varepsilon(x) = \{z\colon x - \varepsilon e \le z \le x + \varepsilon e\}$ is disjoint from $X^+$, and $x$ is an interior point in it. Thus the Hahn–Banach separation theorem guarantees the existence of a constant $c$ with $f(x) < c \le f(w)$ for every $w \in X^+$.

Now $c \le 0$ because $0 \in X^+$. If $f(w) < 0$ for some $w \in X^+$ then $tw \in X^+$ for all $t > 0$, and $f(tw) = tf(w) < c$ if $t$ is large enough. This contradiction shows that $f(w) \ge 0$ for all $w \in X^+$, so we might as well pick $c = 0$.

Now $f(e) > 0$, for we find $-\|z\|f(e) \le f(z) \le \|z\|f(e)$ for all $z$, so if $f(e) = 0$ then $f$ would be the zero functional. Replace $f$ by $f/f(e)$. Then $f(e) = 1$, and it follows that $f \in S$. But this contradicts the assumption that $f(x) \ge 0$ for all $f \in S$.

**Exercise B.6.** The hint was perhaps stated in too complicated a way. Better: Assume that $x \not\le ce$ whenever $c < \|x\|$. (If not, it must be true of $-x$ instead, so we replace $x$ by $-x$.) Thus, whenever $c < \|x\|$ we find $ce - x \not\ge 0$, so there is some $f \in S$ with $f(ce - x) < 0$, i.e., $f(x) > c$. Therefore $\sup_{f \in S}|f(x)| \ge \|x\|$. The opposite inequality is obvious.

**Exercise B.7.** Clearly, $\operatorname{co}(S \cup -S)$ is contained in the (closed) unit ball of $X^*$. Moreover, since $S$ is weakly* compact and convex, then so is $\operatorname{co}(S \cup -S)$: For this is the image of the compact set $[0,1] \times S \times S$ under the continuous map $(t, f, g) \mapsto tf + (1 - t)g$.

If $\operatorname{co}(S \cup -S)$ is not the whole unit ball of $X^*$, pick $h \in X^*$ with $\|h\| \le 1$ and $h \notin \operatorname{co}(S \cup -S)$. By Hahn–Banach separation there is a weakly* continuous functional separating $h$ from $\operatorname{co}(S \cup -S)$. But this then belongs to $X$. I.e., there is some $x \in X$ and a constant $c$ so that $h(x) > c \ge f(x)$ for all $f \in \operatorname{co}(S \cup -S)$. Then $c \ge 0$, and $|f(x)| \le c$ for all $f \in S$. By the previous problem, then $\|x\| \le c$. But this contradicts the inequalities $\|h\| \le 1$, $\|x\| \le 1$, and $h(x) > c$.