Baire category and open mapping

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Abstract. This little note may become a part of my functional analysis notes, later.

Recall that a subset of a metric space is called *rare* if its closure has empty interior. Equivalently, it is contained in a closed set with empty interior. Any countable union of rare sets is called *meager*. Other sets are called *non-meager*.

1 Theorem. (Baire) *Any nonempty complete metric space is non-meager in itself.*

Proof: Let X be a nonempty complete metric space. We only need to show that X is not the union of any countable family of closed sets with empty interior.

The complement (relative to X) of a closed set with empty interior is a *dense, open* set. Thus we only need to show that no countable intersection of dense, open subsets of X is empty.

We shall show a little bit more:

Any countable intersection of dense, open subsets of X is dense.

So let V_k be a dense, open subset of X for $k = 1, 2, \ldots$ and let U be a nonempty, open set.

Since V_1 is dense, we can find some $x_1 \in U \cap V_1$. And because $U \cap V_1$ is open, we can find some $0 < \varepsilon_1 < 1$ so that the *closed* ε_1-ball $\bar{B}_{\varepsilon_1}(x_1)$ around x_1 is contained in $U \cap V_1$.

Next, because V_2 is dense, we can similarly find $x_2 \in B_{\varepsilon_1}(x_1) \cap V_2$, and some $0 < \varepsilon_2 < 1/2$ with $\bar{B}_{\varepsilon_2}(x_2) \subseteq B_{\varepsilon_1}(x_1) \cap V_2$.

Continuing in this way, we find $x_k \in B_{\varepsilon_{k-1}}(x_{k-1}) \cap V_k$, and some $0 < \varepsilon_k < 1/k$ with $\bar{B}_{\varepsilon_k}(x_k) \subseteq B_{\varepsilon_{k-1}}(x_{k-1}) \cap V_k$, for $k = 3, 4, \ldots$.

Now if $m, n \geq k$ then $x_m, x_n \in B_{\varepsilon_k}(x_k)$, so $d(x_m, x_n) < 2\varepsilon_k < 2/k$. Therefore the sequence (x_k) is Cauchy, and thererfore convergent, since X is complete. Let x be its limit.

Since $x_n \in B_{\varepsilon_k}(x_k)$ when $n \geq k$, we find $x \in \bar{B}_{\varepsilon_k}(x_k) \subseteq V_k$. Thus $x \in \bigcap_{k=1}^{\infty} V_k$. But also $x \in \bar{B}_{\varepsilon_1}(x_1) \subseteq U$, so $U \cap \bigcap_{k=1}^{\infty} V_k$ is not empty. Since U was an arbitrary nonempty open set, $\bigcap_{k=1}^{\infty} V_k$ is dense. ∎

If X is a normed space, we write X_1 for its closed unit ball:

$$X_1 = \{x \in X : \|x\| \leq 1\}.$$

Clearly, the closed ε-neighbourhood of any $z \in X$ is $z + \varepsilon X_1$, which by definition is $\{z + \varepsilon x : x \in X_1\}$; and any neighbourhood of z contains $z + \varepsilon X_1$ for some small $\varepsilon > 0$.

It will be useful to note that multiplication by any nonzero scalar is an isomorphism of X to itself, and so it maps open sets to open sets, closed sets to closed sets, etc. We may call this the scale invariance of the topology: For example, if $A \subseteq X$ and $c \neq 0$ then $\overline{cA} = c\overline{A}$, where the overline denotes closure (not to be confused with complex conjugation).

Let X and Y be two normed spaces, and $T\colon X \to Y$ a linear map. We say T is an *open map* if $T(U)$ is open in Y whenever $U \subseteq X$ is open.

2 Lemma. *A linear map $T\colon X \to Y$ is open if, and only if, $T(X_1)$ is a neighbourhood of 0 in Y.*

Proof: If T is open then the image $\{Tx\colon \|x\| < 1\}$ of the *open* unit ball in X is open in Y. Since it contains 0, its superset $T(X_1)$ is a neighbourhood of 0.

Conversely, assume that $T(X_1)$ is a neighbourhood of 0: Then $\varepsilon Y_1 \subseteq T(X_1)$ for some $\varepsilon > 0$.

Let U be an open subset of X. If $u \in U$, pick $\delta > 0$ so that $u + \delta X_1 \subseteq U$. Then $Tu + \delta T(X_1) \subseteq T(U)$, and therefore $Tu + \delta\varepsilon Y_1 \subseteq T(U)$, so that Tu is interior in $T(U)$. Thus every member of $T(U)$ is interior, i.e., $T(U)$ is open. ∎

3 Theorem. (Open mapping) *Any bounded linear mapping of one Banach space onto another Banach space is open.*

Proof: Let $T\colon X \twoheadrightarrow Y$ be a bounded linear map of Banach space X onto Banach space Y. The proof proceeds in several stages.

First note that from $Y = T(X)$ and $X = \bigcap_{k=1}^{\infty} kX_1$ we get $Y = \bigcap_{k=1}^{\infty} kTX_1$.[1] From the Baire theorem, therefore, and the completeness of Y, at least one of the sets kTX_1 is not rare. But by scale invariance that means TX_1 is not rare; i.e., $\overline{TX_1}$ contains an interior point. Say,

$$y_0 + \varepsilon Y_1 \subseteq \overline{TX_1}.$$

Second, pick any $y_1 \in Y_1$. Since y_0 and $y_0 + \varepsilon y_1$ both belong to $\overline{TX_1}$, we find

$$\varepsilon y_1 = (y_0 + \varepsilon y_1) - y_0 \in \overline{TX_1} + \overline{TX_1} = 2\overline{TX_1}.$$

Of course the final equality requires justification, but this is not hard: If (x_k) and (z_k) are two sequences in X_1 so that (Tx_k) and (Tz_k) both converge, then $T(x_k + z_k)$ is a sequence in $2X_1$ which converges to the sum of the two limits.[2]

But this shows that $\varepsilon Y_1 \subseteq 2\overline{TX_1}$, or let us rather say

$$\tfrac{1}{2}\varepsilon Y_1 \subseteq \overline{TX_1}$$

[1] We skip the parentheses in $T(X_1)$ for convenience.

[2] Mopping up the details is left to the reader.

– so we now know that 0 is an interior point of $\overline{TX_1}$.

Third, let us stop for a moment and think about what it means to be in the closure of a subset $A \subseteq Y$: To say that $y \in \overline{A}$ is to say that, for every $\delta > 0$ there is some member of A whose distance to y is at most δ. But *that* is the same as saying $y \in A + \delta Y_1$.

In particular, we now return to the above inclusion, which implies $\frac{1}{2}\varepsilon Y_1 \subseteq TX_1 + \delta Y_1$ for any $\delta > 0$. Let us pick a clever value for δ:

$$\tfrac{1}{2}\varepsilon Y_1 \subseteq TX_1 + \tfrac{1}{4}\varepsilon Y_1. \tag{1}$$

To be very specific about this, let $y_0 \in \frac{1}{2}\varepsilon Y_1$ and use (1) to pick $x_0 \in X_1$ and $y_1 \in \frac{1}{2}\varepsilon Y_1$ with

$$y_0 = Tx_0 + \tfrac{1}{2}y_1.$$

Next, we do the same to y_1, and pick $x_1 \in X_1$ and $y_2 \in \frac{1}{2}\varepsilon Y_2$ with $y_1 = Tx_1 + \frac{1}{2}y_2$. Put together, we have

$$y_0 = Tx_0 + \tfrac{1}{2}Tx_1 + \tfrac{1}{4}y_2.$$

Do the same to y_2 and get

$$y_0 = Tx_0 + \tfrac{1}{2}Tx_1 + \tfrac{1}{4}Tx_2 + \tfrac{1}{8}y_3.$$

By an easy induction, we arrive at

$$y_0 = \sum_{k=0}^{n-1} 2^{-k}Tx_k + 2^{-n}y_n, \qquad x_k \in X_1, \;\; y_n \in \tfrac{1}{2}\varepsilon Y_1.$$

In the end, it is clear that the sum

$$\sum_{k=0}^{\infty} 2^{-k}Tx_k$$

converges, because $\|x_k\| \le 1$ for each k (this is where we use the completeness of X), and if the sum is called x then $\|x\| \le 2$, and in the limit we get $y_0 = Tx$ (and this is where we use the boundedness of T).

Thus we have found that $\frac{1}{2}\varepsilon Y_1 \subseteq 2TX_1$, and that completes the proof that T is open. ∎