75048 Mathematical modeling 1999–01–11

1a Physically meaningful relation written in terms of dimensionless combinations.

1b (i) $V = C\sqrt{Lg}$, and (ii) $V = C\sqrt{\dfrac{\sigma}{\rho L}}$ with C an undetermined constant

1c $L_0 \approx \sqrt{\dfrac{\sigma}{\rho g}} \approx 2.6\,\mathrm{mm}$ (boldly setting the associated constant to 1).

2 Not likely; heat does not penetrate deep enough in the alotted time.

3

$$u_0'' + u_0 = 0, \quad u_1'' + u_1 - 2c_1 u_0'' + u_0^3 = 0, \quad u_0(0) = a, u_0'(0) = u_1(0) = u_1'(0) = 0;$$

$$u_0(\tau) = a\cos\tau, \quad u_1(\tau) = \frac{1}{32}a^3(\cos\tau - \cos 3\tau) - \underbrace{\Big(ac_1 + \frac{3}{8}a^3\Big)\tau\sin\tau}_{\text{secular term}};$$

$$c_1 = -\frac{3}{8}a^2, \quad \text{period} = 2\pi\Big(1 - \frac{3}{8}a^2\varepsilon\Big) + \mathcal{O}(\varepsilon^2).$$

4 $u(x) \approx \dfrac{1}{2-x^2} - \exp\dfrac{x-1}{\sqrt{\varepsilon}}$ to lowest order.

5a

$$C\frac{dT_j^*}{dt^*} = R^*(T_j^*)(I_j^*)^2 - A(T_j^* - T_a) - B(T_j^* - T_{3-j}^*), \quad R^*(T_1^*)I_1^* = R^*(T_2^*)I_2^*, \quad I_1^* + I_2^* = I$$

$$\text{Scale by } I_j^* = II_j, \quad T_j^* = T_a + \Theta T_j, \quad t^* = \frac{C}{A}t; \quad \text{get } \beta = \frac{B}{A}.$$

5b

$$\dot{T}_j = \frac{R(T_j)R(T_{3-j})^2}{\big(R(T_1) + R(T_2)\big)^2} - T_j - \beta(T_j - T_{3-j}) \qquad (j = 1, 2).$$

$$\text{Jacobian } \begin{bmatrix} -(\beta+1) & \beta + R'/4 \\ \beta + R'/4 & -(\beta+1) \end{bmatrix}, \quad \text{eigenvalues } \{-1 + R'/4, -1 - 2\beta - R'/4\}.$$

6a $f^*(u^*) = V \cdot u^* \cdot (1 - u^*/u_{\max})$, scale by $u^* = u_{\max}u$, $t^* = Tt$ and $x^* = Xx$, where $X = VT$.

6b, c **b:** $f(u_r) - f(u_l) = \dot{s} \cdot (u_r - u_l)$; lefthand picture below. **c:** Remaining three pictures.