The damped pendulum

The equation

```
> damped_pendulum:=D(theta)=omega,
  D(omega)=-nu*omega-sin(theta):
  damped_pendulum[1];damped_pendulum[2];
```

$$D(\theta) = \omega$$

$$D(\omega) = -\nu\,\omega - \sin(\theta)$$

```
> dynamics:=subs({theta=theta(t),D(theta)=D(theta)(t),
                  omega=omega(t),D(omega)=D(omega)(t)},
            [damped_pendulum]);
```

$$dynamics := [D(\theta)(t) = \omega(t), D(\omega)(t) = -\nu\,\omega(t) - \sin(\theta(t))]$$

Maple technicalities

Linearized analysis at the equilibrium points

```
> Dpendulum:=linalg[matrix](2,2,eval(subs(`damped_pendulum`,
     '[diff(D(theta),theta),diff(D(theta),omega),
      diff(D(omega),theta),diff(D(omega),omega)]')));
```

$$Dpendulum := \begin{bmatrix} 0 & 1 \\ -\cos(\theta) & -\nu \end{bmatrix}$$

First, the stable equilibrium:
Maple notes: subs will not look inside a structure such as a matrix; hence the use of map.
We must use eval because otherwise $\cos(0)$ is not simplified to 1.

```
> map(x->eval(subs(theta=0,omega=0,x)),Dpendulum);
```

$$\begin{bmatrix} 0 & 1 \\ -1 & -\nu \end{bmatrix}$$

```
> linalg[eigenvals](%);
```

$$-\frac{1}{2}\nu + \frac{1}{2}\sqrt{\nu^2 - 4}, -\frac{1}{2}\nu - \frac{1}{2}\sqrt{\nu^2 - 4}$$

For positive ν this is always stable; a node if $\nu < 2$ and a focus if $2 < \nu$.

```
> map(x->eval(subs(theta=Pi,omega=0,x)),Dpendulum);
```

$$\begin{bmatrix} 0 & 1 \\ 1 & -\nu \end{bmatrix}$$

```
> linalg[eigenvals](%);
```

$$-\frac{1}{2}\nu + \frac{1}{2}\sqrt{\nu^2 + 4}, -\frac{1}{2}\nu - \frac{1}{2}\sqrt{\nu^2 + 4}$$

This is always a saddle point.

Plotting numerical solutions as functions of time

```
> {op(subs(nu=0,dynamics)),theta(0)=0,omega(0)=0.4};
```

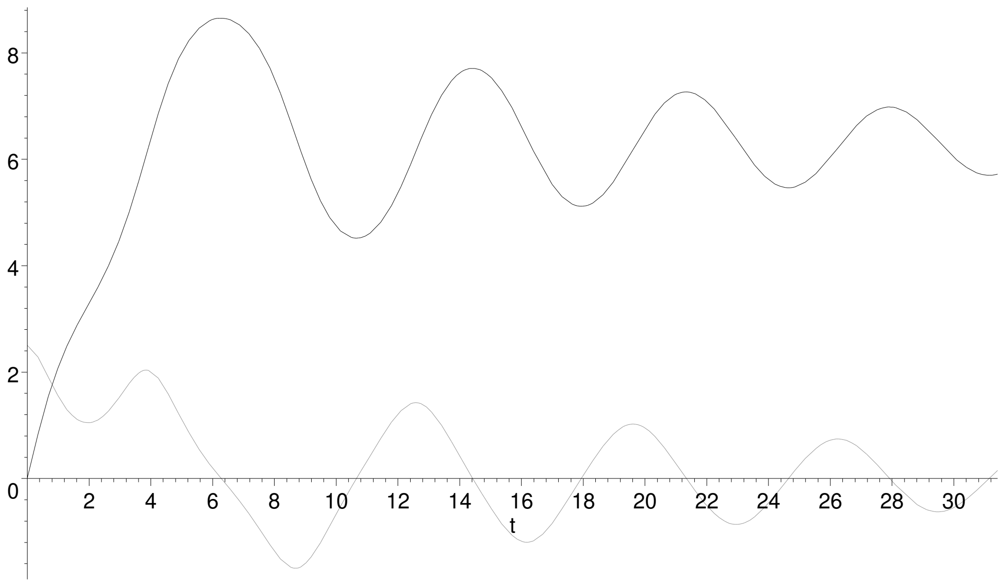
$$\{D(\theta)(t) = \omega(t), \theta(0) = 0, \omega(0) = .4, D(\omega)(t) = -\sin(\theta(t))\}$$

Experiment with the initialvalues below. If $\theta(0) = 0$ and $|\omega(0)| < 2$ with no damping ($\nu = 0$) then the motion is periodic.

```
> initvalues:=theta(0)=0,omega(0)=2.5:
```

```
damping:=nu=0.1:
damping, initvalues;
numsolutions:=dsolve({op(subs(damping,dynamics)),initvalues},
  {theta(t),omega(t)}, type=numeric, output=listprocedure):
nth:=subs(numsolutions,theta(t)):
nom:=subs(numsolutions,omega(t)):
thplot:=plot('nth(t)',t=0..10*Pi,color=darkblue):
omplot:=plot('nom(t)',t=0..10*Pi,color=green):
plots[display](thplot,omplot);
```

$$\nu = .1, \theta(0) = 0, \omega(0) = 2.5$$



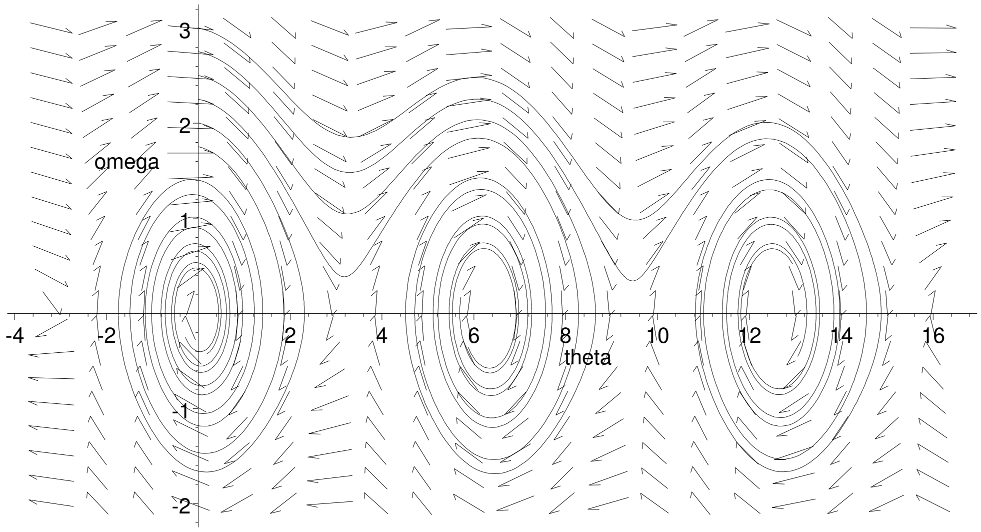

```
>
```

A look in the phase plane

```
> initials:=map(p->[theta(0)=0,omega(0)=p],
    [1.75, 2.0, 2.25, 2.5, 2.75, 3.0]);
  damping:=nu=0.1:
  deplot:=DEtools[DEplot](subs(damping,dynamics),

  [theta,omega],t=0..10*Pi,initials,theta=-3.2..16,omega=-2..3,
           stepsize=0.05,
           colour=black,linecolour=darkblue):
  deplot;
```

$$initials := [[\theta(0) = 0, \omega(0) = 1.75], [\theta(0) = 0, \omega(0) = 2.0], [\theta(0) = 0, \omega(0) = 2.25], [\theta(0) = 0, \omega(0) = 2.5], [\theta(0) = 0, \omega(0) = 2.75], [\theta(0) = 0, \omega(0) = 3.0]]$$



But it seems harder to make Maple plot the separatrices for us. At present, it seems easiest to print out the plot and then do separatrices by hand!