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EXAM IN 75048 MATHEMATICAL MODELING

Monday 11 january 1999
Time: 09:00–15:00

Aids: B1
— Approved calculator, with empty memory
— No printed or written aids allowed

The problem set contains 11 parts.
Only your best ten answers count.

Problem 1

a) What does Buckingham's pi theorem say?

b) For water waves in deep water it may seem reasonable that the wave speed V depends only on the wave length L, the acceleration of gravity g, and the density ρ of the water. What is the most general form of this dependence, if the assumption is correct? Do the same for very short waves (ripples), where we assume that g plays no role, but that the surface tension σ is a deciding factor instead.

c) For a medium short wavelength L_0, gravity and surface tension will influence wave motion to approximately the same extent. Give a rough estimate for L_0.

The surface tension for water is $\sigma \approx 0.07\,\mathrm{N/m}$.

Problem 2 Important parameters in the assessment of foods like meat and fish are water and fat content. For use in quality control there is a need for instruments which quickly and reliably can yield an approximate value for these parameters. In particular, it is an advantage for the test to be non-destructive, so that foods can be tested and then sold. One suggested instrument uses the fact that water, fat and proteins have different thermal properties. The idea is to add a known amount of energy in the form of heat to the surface of the food, and then to measure the resulting increase in the temperature. For this instrument to be useful, it should satisfy the following two criteria:

1. The measurement should not take much more than a half minute.
2. The measurement should give a representative average value for water and fat content not only near the surface, but down to a centimeter or deeper.

Approximate values for the specific capacity of heat and heat conductivity for the most important parts of food are:

			Water	Fat	Protein
Heat capacity	c	$\mathrm{J\,kg^{-1}\,K^{-1}}$	4180	2000	1300
Heat conductivity	k	$\mathrm{J\,m^{-1}\,K^{-1}\,s^{-1}}$	0.56	0.09	0.16

You can assume the density to be $\rho = 1000\,\mathrm{kg\,m^{-3}}$ for all three. Employ a simple dimensional analysis technique to assess whether it would seem probable that such an instrument could be made to fulfill the requirements.

Problem 3 Every solution of Duffing's equation

$$\ddot{u} + u + \varepsilon u^3 = 0 \qquad (\text{with } \varepsilon > 0)$$

is bounded, because $\dot{u}^2 + u^2 + \frac{1}{2}\varepsilon u^4$ is constant. Even so, a naïve attempt at a perturbation expansion leads to unbounded approximate solutions, as we shall see below. The reason is that the solutions are periodic, and we must take into account the fact that the period depends on ε in order to get a generally valid approximation.

Therefore, replace the independent variable t by τ, where $t = (1+\varepsilon c_1 + \mathcal{O}(\varepsilon^2))\tau$. Write up the equations for the first two terms in a resulting perturbation expansion $u = u_0 + \varepsilon u_1 + \mathcal{O}(\varepsilon^2)$, where you assume the initial conditions $u(0) = a$ and $\dot{u}(0) = 0$. Solve these equations for u_0 and u_1, and show that the secular (i.e., unbounded) term in u_1 vanishes if c_1 is chosen correctly. What will the period of the solution be, to first order in ε?

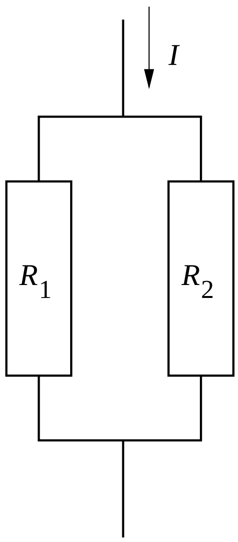
The formula $\cos^3\tau = \frac{1}{4}\cos 3\tau + \frac{3}{4}\cos\tau$ may be useful.

Problem 4 Use singular perturbation to find an approximate solution to

$$\varepsilon u'' - (2 - x^2)u + 1 = 0, \qquad u'(0) = 0, \quad u(1) = 0$$

valid for small ε, and x in the interval $0 \le x \le 1$.

Problem 5 In the production of ferrosilica electrodes are used, which are baked from a carbon paste and then consumed in the furnace in a continuous process. There is reason for concern that the electrode may unevenly baked, because a well-baked part of the electrode conducts electricity much better than an unbaked one. This could lead to most of the electrode current passing through the well-baked region, which then becomes hotter still at the expense of the unbaked part. We shall consider an extremely simplified model which, we hope, casts some light on the basic mechanism without attempting to model the phenomenon in full detail.



We consider two temperature dependent resistors, with a resistance $R^*(T_j^*)$ (for $j = 1, 2$). These are connected in parallel to a constant current source. This will heat up the resistors, which in its turn changes the resistances, and hence voltage and dissipated power. We shall assume that the two resistors are identical, with a heat capacity of C. They lose heat to the surroundings at a rate which is A times the difference between their own temperature and that of the surroundings, and the hotter one loses heat to the cooler one at a rate which is B times the temperature difference between the two. (The heat transfer to the surroundings should perhaps have been given as a radiation law, that is proportional to a difference between fourth powers of temperatures, but this only complicates matters needlessly without adding new insight. Permit me, by the way, to remind you of Ohm's law $U = RI$ and that a current I delivered into a voltage U dissipates a power $P = UI$.)

a) Write equations (with associated reasoning) describing the development of temperature and current in the two resistors, and explain how a suitable rescaling leads to a system on the form

$$\begin{aligned} \dot{T}_j &= R(T_j)I_j^2 - T_j - \beta(T_j - T_{3-j}) \quad (j = 1, 2) \\ R(T_1)I_1 &= R(T_2)I_2 \\ I_1 + I_2 &= 1 \end{aligned}$$

where R is a function, β a constant, and $\dot{T}_j = dT_j/dt$.

b) Assume the above model has a symmetric equilibrium point, that is $\dot{T}_j = 0$ with $T_1 = T_2$. Show that this equilibrium is stable if and only if $-4(1 + 2\beta) < R'(T_1) < 4$. (Here R' is the derivative of the function R. It is the instability for large negative values of R' that is relevant to the problem explained in the introduction. *Hint*: Use the algebraic equations to eliminate I_1 and I_2 from the model, so that you are left with a two-dimensional dynamical system (two first order differential equations). Linearize this around the equilibrium point.

Problem 6 This problem will be based on the model

$$\frac{\partial u}{\partial t} + \frac{\partial f(u)}{\partial x} = 0 \qquad \text{der } f(u) = u - u^2$$

for traffic along a road, where $u(x,t)$ is the density of cars. (The model is hardly quantitatively correct, but qulatitatively it is. It is easy to compute with, and therefore much used.)

a) Briefly explain what assumptions underlie the model, explain how it is derived, and indicate the scaling which will bring it to the above form.

b) What does the Rankine-Hugoniot condition for the speed of a shock say? Sketch characteristics and shocks for the solution of the model with initial data

$$u(x,0) = \begin{cases} \frac{1}{2} & \text{if } 0 < x < 1, \\ \frac{1}{4} & \text{otherwise.} \end{cases}$$

Make sure not to include unphysical shocks in the solution. You do not need to compute the solution in detail in this or the next question.

c) Assume now that the road is initially evenly trafficed with a traffic density of u_0, where $0 < u_0 < \frac{1}{2} - \frac{1}{4}\sqrt{2}$. At the time $t = 0$ a ferry arrives and releases a stream of cars onto the road at $x = 0$. We shall assume that the new traffic is mixed into the old one by merging, so that a stationary discontinuity arises at $x = 0$, with

$$f(u(0+,t)) = 2f(u(0-,t))$$

where $u(0\pm,t)$ are one-sided limits of u at $x = 0$. (This discontinuity does not fulfill the Rankine-Hugoniot condition, since the number of cars is not conserved at $x = 0$. Still, it seems unreasonable to accept solutions with $u(0+,t) < u(0-,t)$.) Sketch characteristics and shocks for a solution of the problem. How does the solution change if $\frac{1}{2} - \frac{1}{4}\sqrt{2} < u_0 < \frac{1}{2}$, or $u_0 > \frac{1}{2}$? (The magical value $\frac{1}{2} - \frac{1}{4}\sqrt{2}$ is a solution to $f(u) = \frac{1}{8}$. The largest value of f is $\frac{1}{4}$.)