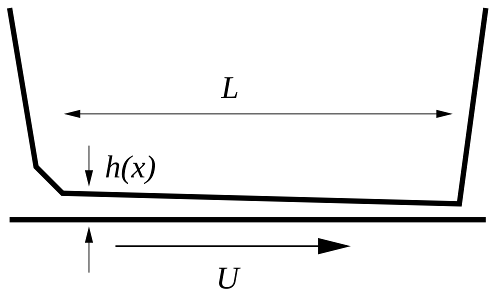
75048 Mathematical modeling 1998

Exercise 3

Advice/Solution: 1998–09–24/1998–10–01



Problem 1: Classical *lubrication theory* is concerned with the flow of a viscous fluid (oil) in a narrow space like between the stationary pad in the figure and a moving surface below, travelling right at the fixed speed U. Assume the length of the pad is L and the (varying) width of the oil filled gap is $h^* \approx \varepsilon L$, where ε is a small constant. The steady-state Navier-Stokes and continuity equations for an incompressible fluid filling the gap are

$$\begin{aligned} u^*\partial_{x^*}u^* + v^*\partial_{y^*}u^* &= -(\partial_{x^*}p^*)/\rho + \nu(\partial_{x^*x^*}u^* + \partial_{y^*y^*}u^*) \\ u^*\partial_{x^*}v^* + v^*\partial_{y^*}v^* &= -(\partial_{y^*}p^*)/\rho + \nu(\partial_{x^*x^*}v^* + \partial_{y^*y^*}v^*) \\ \partial_{x^*}u^* + \partial_{y^*}v^* &= 0 \end{aligned}$$

where we have written * on quantities that will later be scaled, and used the notation ∂_{x^*} for $\partial/\partial x^*$. u^* and v^* are the x^* and y^* components of the fluid velocity. The boundary conditions are

$$u^* = U, v^* = 0 \text{ at } y^* = 0, \quad u^* = v^* = 0 \text{ at } y^* = h^*(x^*)$$

(a) Explain why the following formulas give reasonable scalings of horizontal and vertical lenghts and velocities respectively:

$$x^* = Lx, \quad y^* = \varepsilon L y, \quad u^* = Uu, \quad v^* = \varepsilon U v$$

(Be careful with the last one, which is less obvious than the rest).

(b) It is less clear how to scale pressure, so just put $p^* = Pp$ where P is an unknown pressure scale, and write up the scaled problem. Assuming that the pressure term containing $\partial_{x^*}p^*$ balances the dominant part of the viscous term, suggest a value for P and write the resulting equation in terms of ε and Reynold's number $\mathrm{Re} = \varepsilon LU/\nu$.

(c) Estimate Reynold's number for a typical case where $L \approx 20\,\mathrm{mm}$, $\varepsilon L \approx 10\,\mu\mathrm{m}$, $U \approx 10\,\mathrm{m/s}$. The kinematic viscosity of lubrication oils is usually at most a few centiStokes, where Stokes is the natural CGS unit for kinematic viscosity (that is, cm^2/s). Turbulence can be expected when Reynold's number exceeds 1000 or so. Hopefully, we can conclude that this will not happen here.

(d) To lowest order, i.e., setting $\varepsilon = 0$ in the equations, you should find

$$\partial_x p = \partial_{yy}u, \quad \partial_y p = 0, \quad \partial_x u + \partial_y v = 0 \tag{1}$$

If not, go back and try to discover what went wrong. Show that

$$u = -\frac{1}{2}(\partial_x p)y(h-y) + \frac{h-y}{h} \tag{2}$$

(Hint: Integrate the first equation of (1) twice in the y direction). Also show that $\int_0^{h(x)} \partial_x u = 0$ by integrating the third equation of (1) in the y direction, and combine with (2) to obtain $\frac{1}{6}\partial_x(h^3\partial_x p) = \partial_x h$, which can be integrated again, resulting in *Reynold's equation*:

$$\frac{\partial p}{\partial x} = 6\frac{h - \overline{h}}{h^3}$$

where the integration constant $\overline{h}$ needs to be determined from suitable boundary conditions. Do this, assuming $p(0) = p(1) = 0$ and $h(x) = 1 - \alpha x$.

Problem 2: Inspired by the previous problem, "long and thin" models are often solved by similar techniques, called "lubrication theory". Here we shall look at heat conduction in a long thin two-dimensional bar, of length L and width $2h$, where $\varepsilon = h/L$ is small. The two ends (at $x^* = 0$ and $x^* = L$) are held at fixed temperatures T_0^* and T_1^*, while along the sides (at $y^* = \pm h$) heat is leaking out at a rate proportional to $T^* - T_0^*$.

(a) Show that the problem becomes

$$\partial_{xx}T + \varepsilon^{-2}\partial_{yy}T = 0, \quad T(0,y) = 0,\ T(1,y) = 1,\ \partial_y T \pm (\alpha\varepsilon)^2 T = 0 \text{ at } y = \pm 1$$

after suitable scaling.

(b) Write $T = T_0 + \varepsilon^2 T_1 + O(\varepsilon^4)$ in these equations. From the $O(1)$ terms, show that T_0 depends on x only, but is otherwise undetermined. Then, by looking at the $O(\varepsilon^2)$ terms, find out as much as possible about T_1 and use this to show that $T_0 = \sinh(\alpha x)/\sinh\alpha$.

Problem 3 (Exam, August1996, somewhat modified): A sprinter's top speed is, among other things, determined by the maximum force e[1] is capable of producing. Write this as MP, where M is the mass of the sprinter and P the maximum "acceleration". Furthermore, the runner's "inner resistance"[2] at speed u^* acts as a counter force which can be approximated by the expression Mu^*/τ, where τ has the dimension of time. Measurements on various runners, among them Ben Johnson and Carl Lewis, yielded $P \approx 10\,\mathrm{m/s^2}$ and $\tau \approx 1\,\mathrm{s}$.

(a) Explain why the runner's equation of motion in still air may be written

$$M\frac{du^*}{dt^*} + M\frac{u^*}{\tau} + \frac{1}{2}\rho_{\mathrm{air}} C_d A (u^*)^2 = M p^*(t^*)$$

and briefly explain how the expression for the air resistance arises from dimensional analysis (where C_d is a function of the Reynold's number, and A is the runner's cross sectional area in the direction of motion).

(b) Find suitable scales for u^*, t^*, and p^* so that the equation in dimensionless form becomes

$$\dot{u}(t) + u(t) + \varepsilon u(t)^2 = p(t), \qquad \text{where} \quad \varepsilon = \frac{1}{2}\rho_{\mathrm{air}} C_d \frac{A}{M}\tau^2 P.$$

Does air resistance play a large role? Assume $C_d \approx 1$.

(c) Assume the sprinter uses maximum power the whole distance, i.e., $p^* = P$. Determine the first two terms in a perturbation expansion for u in this case. How long does it take for the runner to reach approximately top speed?

(d) It is well known that the wind can be an important factor in the 100 m run. Show how the equation of motion is changed if a wind of speed W blows parallel to the direction of the run. Find, to order ε, the highest speed a runner can be expected to manage as a function of $\delta = W/(\tau P)$.

(e) During the qualification for the olympic games in 1988, Florence Griffith-Joyner broke the 100 m world record with a 10.49 s run. In the other runs, and during the games, she achieved times around 10.7 s. The wind gauge registered still air during the record run, but many people present argued that there was a considerable tail wind (at $\approx 4\,\mathrm{m/s}$) and that the wind gauge did not work. Discuss this claim.

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[1] He or she.

[2] A combination of the effects of acceleration and friction in arms and legs.