75048 Mathematical modeling 1998

Exercise 6

Advice/Solution: 1998–11–05/1998–11–12

Problem 1 (Exam 1990–12): Consider a viscous fluid with viscosity μ running down an incline with slope α. Let the x axis lie along the incline with the positive direction pointing downhill, and the y axis be normal to the incline and directed upwards, with $y = 0$ on the surface of the incline. Assume at first that the liquid flows in a layer of constant thickness h (so that $y = h$ at the surface of the liquid), and that the liquid velocity v in every point is a function of y only, and is directed along the x axis. The shear stress acting across a section through the liquid parallel to the incline is $\mu\, dv/dy$.

a. Show that dv/dy is proportional to $h - y$. (*Hint*: Pick a material volume between a given height y and the surface, and note that the acceleration is zero.) Then show that v is proportional to $2hy - y^2$ and find the factor of proportionality. What extra physical assumption is needed to reach this conclusion? Finally, find an expression for the total liquid flow rate $Q(h)$ past a fixed point on the incline.

b. We shall now assume that the formula found above for $Q(h)$ is also valid when h is not constant. Write up a conservation law for the fluid on integral form. Rename the variables h, x, t, and the flux function to h^*, x^*, etc., and show that for any scales $T > 0$ and $H > 0$ we can find $X > 0$ so that the scaling $t^* = Tt$, $h^* = Hh$, and $x^* = Xx$ transforms the conservation law into

$$\partial_t h + \partial_x Q(h) = 0, \qquad Q(h) = \frac{1}{3}h^3 \tag{1}$$

on differential form.

We shall solve (1) in the region $x > 0$, $t > 0$ with the initial and boundary data

$$h(x,0) = 0, \quad h(0,t) = \sqrt{t}.$$

c. Find the equation of characteristics starting at the t axis, and show that a point (x,t) is on the characteristic from $(0,t_0)$ if and only if $t_0 = t/2 \pm \sqrt{t^2/4 - x}$.

d. Assume that a shock is formed, where each point (x,t) is hit from the left by a characteristic starting in $(0, t/2 + \sqrt{t^2/4 - x})$, and from the right by a characteristic from the x axis. Find an expression for the shock speed, and use this to determine a differential equation for the shock position $s(t)$. Fortunately, there is no need to find a general solution to this equation! Find a constant α so that $s(t) = \alpha t^2$ solves the equation. Compare the shock speed with the speed of the two characteristics from $(0, t/2 \pm \sqrt{t^2/4 - x})$, and verify that one of them (the '+' one) really reaches the shock at $(s(t), t)$, while the other one does not. Sketch the characteristics and the shock in the (x,t) plane.

Problem 2 (Exam 1985–01): The students' 10 km run is arranged in a stadion with a circular track of length L ($= 400$ m). We shall assume that the running speed v^* decreases linearly with density, and that $v^* = v^*_{\max}$ when $\rho^* = 0$ and $v^* = 0$ when $\rho = \rho_{\max}$.

a. Write up a conservation law for students under the assumptions above, introduce dimensionless variables, and show that the differential formulation can be written on the form

$$\partial_t \rho + (1 - 2\rho)\partial_x \rho = 0 \tag{2}$$

where $0 \le \rho \le 1$ and $\rho(x,t)$ is a 2π–periodic function of x.

b. Give (in implicit form) the exact solution of (2) if

$$\rho(x,0) = \rho_0 + \varepsilon \cos x \qquad (0 < \rho_0 \pm \varepsilon < 1, \varepsilon > 0)$$

c. Sketch the characteristics corresponding to the initial values in b. Show that the solution must break down after some time and develop a shock.

d. When is the shock formed, how does it move, and what happens as $t \to \infty$? (*Hint*: consider the point of intersection of characteristics starting at $3\pi/2 + \theta$ and $3\pi/2 - \theta$, where $0 < \theta < \pi$.)