75048 Mathematical Modeling 1998

Exercise 4 – Solution

October 18, 1998

Problem 1:

We write $y_o = y_0 + \varepsilon y_1 + \cdots$ and $Y_i = Y_0 + \varepsilon Y_1 + \cdots$ for the outer and inner approximation respectively. The uniform approximation is denoted by y_u.

a)

Substituting y_o into the equation and collecting terms of the same power of ε gives

$$(1+x)y_0' + y_0 + \varepsilon(y_0'' + (1+x)y_1' + y_1) = \mathcal{O}(\varepsilon^2), \tag{1}$$

leading to two first order equations for y_0 and y_1. We therefore only need one boundary condition (that is not in the boundary layer). It will be apparent which one we are to use from the solution. The equation for y_0 has general solution $y_0 = C/(1+x)$. If we use the boundary condition $y(0) = 0$ to determine C we get $y_0 \equiv 0$. Below we will show that it is impossible to match this solution with the inner approximation, and we therefore use the boundary condition at $x = 1$ and obtain

$$y_0(x) = \frac{2}{1+x}. \tag{2}$$

To determine the inner approximation we scale x such that the new variable varies roughly between 0 and 1 in the boundary layer, $\xi = x/\varepsilon$, which upon substitution in the original equations yields

$$\frac{d^2y}{d\xi^2} + \frac{dy}{d\xi} + \varepsilon(\xi\frac{dy}{d\xi} + y) = 0. \tag{3}$$

Substituting Y_i and collecting terms gives

$$\ddot{Y}_0 + \dot{Y}_0 + \varepsilon(\ddot{Y}_1 + Y_1 + \xi\dot{Y}_0 + Y_0) = \mathcal{O}(\varepsilon^2), \tag{4}$$

where derivation is with respect to ξ. We assume the solution is valid close to $\xi = 0$ and therefore use the boundary condition $Y_i(0) = 0$, and thus $Y_0(0) = Y_1(0) = 0$. We obtain $Y_0 = K(1 - e^{-\xi})$.
To obtain the uniform approximation we first require

$$\lim_{\varepsilon\to 0} Y_i(\eta\theta(\varepsilon)/\varepsilon) = \lim_{\varepsilon\to 0} y_o(\eta\theta(\varepsilon)),$$

which gives $K = 2$ and

$$y_u(x) = y_o(x) + Y_i(x/\varepsilon) - 2 = \frac{2}{1+x} - 2e^{-x/\varepsilon}.$$

If we had used the condition $y_o(0) = 0$ we would have obtained $y_o = 0$ and $y_u = 0$ which is clearly wrong.

b)

The equation determining y_1 is

$$y_0'' + (1+x)y_1' + y_1 = 0, \quad y_1(1) = 0,$$

and after using the expression for y_0 we obtain (using $((1+x)y_1)' = (1+x)y_1' + y_1$)

$$y_1(x) = -\frac{1}{2}(1+x)^{-1} + 2(1+x)^{-3}.$$

Similarly Y_1 is determined from

$$\ddot{Y}_1 + \dot{Y}_1 + \xi\dot{Y}_0 + Y_0 = 0, \quad Y_1(0) = 0.$$

This equation has solution

$$Y_1(\xi) = \xi^2 e^{-\xi} - 2\xi - Ae^{-\xi} + B,$$

where the boundary condition gives $A = B$. Setting $A = C + 2$ produces

$$Y_1(\xi) = C(1 - e^{-\xi}) - 2(e^{-\xi} + \xi - 1 - \frac{1}{2}\xi^2 e^{-\xi}),$$

and we are done.

c)

As above we set $x = \theta(\varepsilon)\eta(\varepsilon) = \theta\eta$. In the outer solution this gives

$$\begin{aligned}
y_o(\theta\eta) &\approx y_0(\theta\eta) + \varepsilon y_1(\theta\eta) \\
&= \frac{2}{1+\theta\eta} + \varepsilon\left(\frac{2}{(1+\theta\eta)^3} - \frac{1}{2}\frac{1}{1+\theta\eta}\right) \\
&= 2(1 - \theta\eta + \mathcal{O}((\theta\eta)^2)) + \varepsilon(2(1 - 3\theta\eta + \mathcal{O}((\theta\eta)^2)) - \frac{1}{2}(1 - \theta\eta + \mathcal{O}((\theta\eta)^2))) \\
&= 2 - 2\theta\eta + \frac{3}{2}\varepsilon + \mathcal{O}((\theta\eta)^2) + \mathcal{O}(\varepsilon\theta\eta).
\end{aligned}$$

For the inner solution we obtain

$$\begin{aligned}
Y_i\left(\frac{\eta\theta}{\varepsilon}\right) &\approx Y_0\left(\frac{\eta\theta}{\varepsilon}\right) + \varepsilon Y_1\left(\frac{\eta\theta}{\varepsilon}\right) \\
&= 2(1 - e^{-\theta\eta/\varepsilon}) + \varepsilon C(1 - e^{-\theta\eta/\varepsilon}) - 2\varepsilon(1 - e^{-\theta\eta/\varepsilon} + \frac{\theta\eta}{\varepsilon} - 1 - \frac{1}{2}\left(\frac{\theta\eta}{\varepsilon}\right)^2 e^{-\theta\eta/\varepsilon}) \\
&= 2 - 2\theta\eta + (C+2)\varepsilon + \text{TST},
\end{aligned}$$

and comparison gives $C = -1/2$.

d)

The inner and outer approximation has the common term $2 - 2x + \frac{3}{2}\varepsilon$, and thus

$$\begin{aligned} y_u(x) &\approx y_0(x) + \varepsilon y_1(x) + Y_0(\tfrac{x}{\varepsilon}) + \varepsilon Y_1(\tfrac{x}{\varepsilon}) - (2 - 2x + \tfrac{3}{2}\varepsilon) \\ &= (\frac{2}{1+x} - 2e^{-x/\varepsilon}) + \varepsilon(\frac{2}{(1+x)^3} - \tfrac{1}{2}\frac{1}{1+x} - \tfrac{3}{2}e^{-x/\varepsilon} + \frac{x^2}{\varepsilon^2}e^{-x\varepsilon}). \end{aligned}$$

Problem 2:

a)

We choose a control volume R^* and set up the following conservation law for the amount of substrate inside R^*:

$$\frac{d}{dt^*}\int_{R^*} s^* d^3\mathrm{x}^* - \int_{\partial R^*} D^*\nabla s^* \cdot \mathrm{n} d\sigma^* = \int_{R^*} k_1 \bar{e}\,\bar{s}[-s + (s+\kappa-\lambda)c]d^3\mathrm{x}^*$$

The surface integral on the left side is the flux across the boundary caused by diffusion and the integral on the right is the amount of substrate that is formed per time unit within R^* (because we are working in the original units we get the factor $k_1\bar{e}\,\bar{s}$). Check the text book if something is not clear.

We use the given scales in the equation above: [1]

$$k_1\bar{e}\frac{d}{dt}\int_R \bar{s}sa^3d^3\mathrm{x} - \int_{\partial R} D^*\bar{s}a^{-1}\nabla s\cdot \mathrm{n}a^2 d\sigma = \int_R k_1\overline{es}[-s + (s+\kappa-\lambda)c]a^3d^3\mathrm{x}$$

or,

$$\frac{d}{dt}\int_R sd^3\mathrm{x} - \int_{\partial R} D\nabla s\cdot \mathrm{n}d\sigma = \int_R [-s + (s+\kappa-\lambda)c]d^3\mathrm{x}, \quad D = \frac{D^*}{k_1}lineea^2.$$

By the divergence theorem and by changing the order of integration and derivation we find

$$\int_R s_t d^3\mathrm{x} - \int_R D\nabla^2 s d\sigma = \int_R [-s + (s+\kappa-\lambda)c]d^3\mathrm{x},$$

and by Dubois-Reymond's lemma we obtain $s_t - D\nabla^2 s = -s + (s+\kappa-\lambda)c$. The equation for c_t can be found in the same way, but the derivation is much easier as there is no flux term to take into consideration.

b)

Under stationary conditions we have $c_t = s_t = 0$, and from the equation for c_t we get that $c = s/(s+\kappa)$. Substituting this into the equation for s_t, and using the fact that $s_t = 0$ yields

$$0 = s_t = D\nabla^2 s - s + (s+\kappa-\lambda)c = D\nabla^2 s - \frac{\lambda s}{s+\kappa}.$$

[1] When we make a change of variables from x^* to x then ∇ must be replaced by $a^{-1}\nabla$. One should really write ∇^* for the gradient operator in the coordinates x^*, so that we can say $\nabla^* = a^{-1}\nabla$

Because of the symmetry in the problem s is a function of r alone. Because $\nabla^2 s = r^{-2}(r^2 s')'$ in this case we can write the equation above as

$$D(r^2 s')' = \frac{\lambda r^2 s}{s+\kappa}. \tag{5}$$

The condition $s = 1$ for $|\mathrm{x}| = 1$ gives the boundary condition $s(1) = 1$. Since we are using spherical coordinates we must also have that $s(r)$ is bounded as $r \to 0$. If this is not clear now it will become clear as we solve the equations below.

The total reaction rate at stationary conditions is equal to the total flux of substrate crossing the particle's boundary:

$$\int_{\mathrm{x}^*=1} D^* \nabla s^* d\sigma^* = \int_{\mathrm{x}=1} D^* a^{-1} \bar{s} \nabla s \cdot \mathrm{n} a^2 d\sigma = 4\pi a D^* \bar{s} s'(1)$$

(or, in dimensionless units, $4\pi D s'(1)$, which is also an acceptable answer, although it is not that readily useful.)

c)

Write (5) as

$$(r^2 s')' = \frac{\varepsilon r^2 s}{s+\kappa}. \tag{6}$$

For $\varepsilon = 0$ this is simply $(r^2 s')' = 0$ giving $r^2 s' = C_1$, $s' = C_1 r^{-2}$, $s = -C_1 r^{-1} + C_2$. The boundary condition in $r = 0$ (s bounded) gives $C_1 = 0$, and $s(1) = 1$ gives $C_2 = 1$, and we get the solution

$$s_0(r) = 1.$$

We now substitute $s(r) = 1 + \varepsilon s_1 + \mathcal{O}(\varepsilon^2)$ into the right hand side in (6):

$$\begin{aligned}\frac{\varepsilon r^2 s}{s+\kappa} &= \varepsilon r^2 (1 - \frac{\kappa}{s+\kappa}) = \varepsilon r^2 (1 - \kappa \frac{1}{1+\kappa+\varepsilon s_1 + \mathcal{O}(\varepsilon^2)}) \\ &= \varepsilon r^2 (1 - \kappa (\frac{1}{1+\kappa} + \frac{\varepsilon s_1 + \mathcal{O}(\varepsilon^2)}{(1+\kappa)^2} + \mathcal{O}(\varepsilon^2))) = \varepsilon \frac{r^2}{1+\kappa} - \varepsilon^2 \frac{r^2 \kappa s_1}{(1+\kappa)^2} + \mathcal{O}(\varepsilon^3)\end{aligned}$$

Now we also have $(r^2 s')' = \varepsilon (r^2 s_1')' + \varepsilon^2 (r^2 s_2')' + \mathcal{O}(\varepsilon^3)$, so (6) gives

$$\begin{aligned}(r^2 s_1')' &= \frac{r^2}{1+\kappa} \\ (r^2 s_2')' &= \frac{r^2 \kappa s_1}{(1+\kappa)^2}\end{aligned}$$

The boundary conditions are: $s_k(1) = 0$ for $k \geq 1$ and $s_k(r)$ bounded as $r \to 0$.

We solve for s_1 first and then for s_2:

$$
\begin{aligned}
r^2 s_1' &= \frac{1}{1+\kappa}\left(\frac{1}{3}r^3 + C_3\right) \\
s_1' &= \frac{1}{1+\kappa}\left(\frac{1}{3}r + C_3 r^{-2}\right) \\
s_1 &= \frac{1}{1+\kappa}\left(\frac{1}{6}r^2 - C_3 r^{-1} + C_4\right) \qquad (C_3 = 0,\ C_4 = -1/6) \\
s_1 &= \frac{r^2-1}{6(1+\kappa)} \\
(r^2 s_2')' &= \frac{\kappa}{6(1+\kappa)^3}(r^4 - r^2) \\
r^2 s_2' &= \frac{\kappa}{6(1+\kappa)^3}(\frac{1}{5}r^5 - \frac{1}{3}r^3 + C_5) \\
s_2' &= \frac{\kappa}{6(1+\kappa)^3}(\frac{1}{5}r^3 - \frac{1}{3}r + C_5 r^{-2}) \\
s_2 &= \frac{\kappa}{6(1+\kappa)^3}(\frac{1}{20}r^4 - \frac{1}{6}r^2 - C_5 r^{-1} + C_6) \qquad (C_5 = 0, C_6 = 1/6 - 1/20) \\
s_2 &= \frac{\kappa}{360(1+\kappa)^3}(3r^4 - 10r^2 + 7)
\end{aligned}
$$

So

$$s = 1 + \frac{\varepsilon}{6(1+\kappa)}(r^2-1) + \frac{\varepsilon^2\kappa}{360(1+\kappa)^3}(3r^4 - 10r^2 + 7) + \mathcal{O}(\varepsilon^3).$$

d)

In this case we write (5) as

$$\eta\left(s'' + \frac{2}{r}s'\right) = \frac{s}{s+\kappa}. \tag{7}$$

We get the outer solution by setting $\eta = 0$, and this gives $s = 0$. For the inner solution we set $s(1-\delta\rho) = \sigma(\rho)$, such that (7) becomes

$$\frac{\eta}{\delta^2}\sigma'' - \frac{2\eta}{\delta - \delta^2\rho}\sigma' = \frac{\sigma}{\sigma+\kappa}$$

If we scale such that σ'' and σ' both become of order 1 the first term on the left will become much larger than the second term as long as δ is small. We therefore have to balance the first term on the left side and the right side. Thus $\delta = \sqrt{\eta}$ is a good choice leading to the equation

$$\sigma'' - \frac{2\sqrt{\eta}}{1-\sqrt{\eta}\rho}\sigma' = \frac{\sigma}{\sigma+\kappa}.$$

To the lowest order in η we get the inner solution by substituting $\eta = 0$ into the equation above, and this gives

$$\sigma''\sigma' = \frac{\sigma}{\sigma+\kappa}\sigma'.$$

An indefinite integral of $\sigma/(\sigma+\kappa)$ with respect to. σ is $\sigma - \kappa\ln(\sigma+\kappa)$, so the equation can be integrated once to give

$$\frac{1}{2}(\sigma')^2 = \sigma - \kappa\ln(\sigma+\kappa) + C_1$$

for some constant C_1. However, we need $\sigma \to 0$ as $\rho \to \infty$ (for the inner and outer solution to match), and the we must also have $\sigma' \to 0$. So the right hand side in the equation above has to become 0 when $\sigma = 0$, such that $C_1 = \kappa \ln\kappa$ and we get

$$\sigma' = -\sqrt{2\sigma - 2\kappa\ln\left(1+\frac{\sigma}{\kappa}\right)}$$

(we must pick the negative square root because σ is to decrease from 1 to 0 as ρ increases from 0 to ∞). Because $\ln(1+u) < u$ for all $u > 0$ the radical(??) is nonnegative, and because of the boundary condition $\sigma(0) = 1$ we get

$$\int_\sigma^1 \frac{d\tilde{\sigma}}{\sqrt{2\tilde{\sigma} - 2\kappa\ln\left(1+\tilde{\sigma}/\kappa\right)}} = \int_0^\rho d\tilde{\rho} = \rho.$$

In the limit $\sigma \to 0$ we can use $\ln(1+u) = u - u^2/2 + \mathcal{O}(u^3)$ to see that the integrand on the left hand side is approximately

$$\frac{\sqrt{\kappa}}{\tilde{\sigma}}$$

such that $\rho \to \infty$ as $\sigma \to 0$, or to state it the other way around, $\sigma \to 0$ as $\rho \to \infty$. The convergence is exponential because the integral goes as $\ln\sigma$.

Because the outer solution is 0 the inner solution is the full approximation to the exact solution, and we have

$$s(r) \approx \sigma\left(\frac{1-r}{\sqrt{\eta}}\right).$$

Problem 3:

System:

$$\frac{dx^*}{dt^*} = r_1 x^*\Big(1 - \frac{x^*}{K} - \frac{b_1 c_1 y^*}{b_1 x^* + c_1}\Big)$$
$$\frac{dy^*}{dt^*} = r_2 y^*\Big(1 - \frac{y^*}{b_2 x^*}\Big)$$

a)

The predator population is at an equilibrium when it equals $b_2 x^*$.

The predator population's ability to catch prey is given by

$$f = r_1 c_1 y^* \frac{b_1 x^*}{b_1 x^* + c_1}.$$

When the prey population x^* is small the number of animals caught will roughly be proportional to the number of (binary) collisions between predator and prey: $f \sim k y^* x^*$. When the prey population x^* is large almost all predators will be able to catch a prey and $f \sim \tilde{k} y^*$.

A typical time scale is the inverse of one of the growth rates. We pick r_1, i.e., $t = r_1 t^*$. (Ignoring the upper limit for the population and also ignoring the f term gives $\frac{dx^*}{dt^*} = r_1 x^*$ with solution $x^*(t^*) = x_0^* \exp(r_1 t^*)$). The prey population has K as its natural scale. As explained above $b_2 K$ then becomes the natural scale for the predator population. This leads to

$$\frac{dx}{dt} = f(x,y) = x\Big(1 - x - \frac{\alpha y}{x+\beta}\Big)$$
$$\frac{dy}{dt} = g(x,y) = \varepsilon y\Big(1 - \frac{y}{x}\Big),$$

where $\varepsilon = r_2/r_1$, $\alpha = c_1 b_1$ and $\beta = c_1/b_1 K$.

b)

The set of equations $f(x,y) = g(x,y) = 0$ has solutions $(0,0)$, $(1,0)$, and (x_0, x_0) in the first quadrant, and where x_0 is the nonnegative solution of $x^2 + (\alpha + \beta - 1)x - \beta = 0$. We have

$$1 - x_0 = \alpha \frac{x_0}{x_0 + \beta},$$

and $0 < x_0 < 1$. We can be more precise:

- If $\alpha < 1$ then $1 - x_0 < \alpha$ and $1 - \alpha < x_0 < 1$.
- If $\alpha > 1$ then $\alpha \frac{x_0}{x_0+\beta} < 1$ leading to $x_0 < \frac{\beta}{\alpha - 1}$.
- If $\alpha = 1$ then $x_0 = x_0(\beta)$ where $x_0(\beta) \sim \sqrt{\beta}$ for small β and $x_0(\beta) \to 1$ when $\beta \to \infty$.

The linearized version of the system above is

$$\begin{bmatrix} \dot{x} \\ \dot{y} \end{bmatrix} = \begin{bmatrix} f_x(x_0, y_0) & f_y(x_0, y_0) \\ g_x(x_0, y_0) & g_y(x_0, y_0) \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix}.$$

where

$$a = 1 - x - \frac{\alpha y}{x+\beta} + x\Big(-1 + \frac{\alpha y}{(x+\beta^2)}\Big),$$
$$b = -\frac{x\alpha}{x+\beta},$$
$$c = \varepsilon \frac{y^2}{x^2},$$
$$d = \varepsilon\Big(1 - 2\frac{y}{x}\Big).$$

Normally we could now substitute values for the singular point (x_0, y_0) and consider the linearized system to determine the system behavior, that is, consider the eigenvalues given by

$$\lambda = -\frac{1}{2}p \pm \frac{1}{2}\sqrt{\Delta}$$

where $\Delta = p^2 - 4q$, $p = -(a+d)$ and $q = ad - bc$. However, the point $(0,0)$ is slightly problematic. Various tricks can be applied, and the end of the story is that $(0,0)$ is a saddle point.

In $(1,0)$ we have $a = -1$, $b = -\frac{\alpha}{1+\beta}$, $c = 0$ and $d = \varepsilon$. Thus, $p = \varepsilon - 1$, $q = -\varepsilon$ and $\Delta = (\varepsilon + 1)^2$. Therefore $(1,0)$ is a *saddle point.* Note that this point is a little special since our system does not have nonnegative y values.

It is harder to say something about the last singular point – the expressions become slightly involved as long as ε, α and β can vary freely. Some cases can be discussed, though. Here

$$\begin{aligned}
a &= x_0\Big(-1 + \frac{\alpha x_0}{(x_0 + \beta^2)}\Big) = -x_0 + \frac{1}{\alpha}\big(1 - x_0\big)^2 = x_0\Big(\frac{1 - x_0}{x_0 + \beta}\Big) - x_0, \\
b &= -\frac{x_0 \alpha}{x_0 + \beta}, \\
c &= \varepsilon, \text{and} \\
d &= -\varepsilon,
\end{aligned}$$

so $p = \varepsilon - a$ and

$$q = \varepsilon x_0 \frac{x_0^2 + 2x_0\beta + \beta^2 + \alpha\beta}{(x_0 + \beta)^2} > 0.$$

Consequently, (x_0, y_0) is not a saddle point. For ε sufficiently large we have $\Delta = (\varepsilon + a)^2 - 4\varepsilon b > 0$ and $p = \varepsilon - a > 0$ and we get a stable node.

We also see that if $x_0 - x_0(\frac{1-x_0}{x_0+\beta}) \geq 0$ the singular point will at least be stable. This happens if $x_0 > \frac{1-\beta}{2}$. Suppose $\alpha < 1$. We know $1 - x_0 < \alpha$ and the condition $x_0 > \frac{1-\beta}{2}$ is satisfied if $\alpha < (1 + \beta)/2$. We conclude that the singular point is stable for $\alpha < 1/2$ for all choices of ε and β.

If $x_0 - x_0(\frac{1-x_0}{x_0+\beta}) < 0$ the singular point will be stable for large values of ε and unstable for small values.

c)

For the parameters given in the problem ($\alpha = 2$, $\varepsilon = 0.2$ and $\beta = 0.15$) we get the singular point $(0.118, 0.118)$. Here $p = -0.070$, $q = 0.12$ and $\Delta = -0.48$ and we conclude that it is an *unstable focus.*

In figure 1 we have drawn a phase portrait of the solution for the given parameters. We clearly see the saddle point in $(1,0)$ and the indicated behavior close to the origin. A more interesting observation is the existence of a stable limit cycle. To prove the existence we need more than linear stability theory.

From the analysis above it follows that if $\beta \to 0$ while ε and α are kept fixed then $x_0 \to 0$.

If, for example, β is increased to 0.25 then p becomes positive and we get a *stable focus.* The system behavior has undergone a significant change (bifurcation). Maple comes in handy and gives us $p(\beta) = 0$ for $\beta \approx 0.2138$. In figure 2 we have sketched a portion of the phase portrait for the values $\beta = 0.15$, $0.2139\ldots$ and 0.25. To get a correct phase portrait for $p = 0$ is hard because of numerical inaccuracies in both Maple and Matlab. Remember that the linear stability analysis (Hartmann-Grobman's theorem) does not apply for centers.

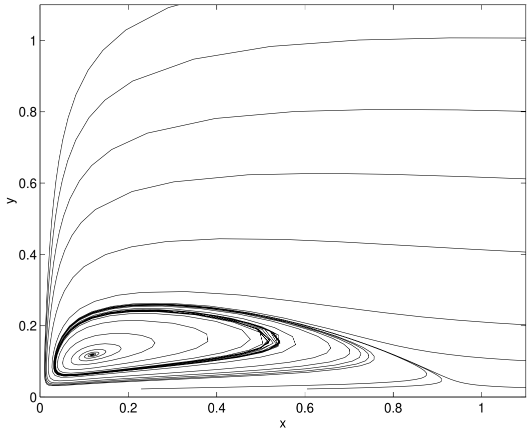

Figure 1: Phase portrait for $\alpha = 2$, $\varepsilon = 0.2$ and $\beta = 0.15$.

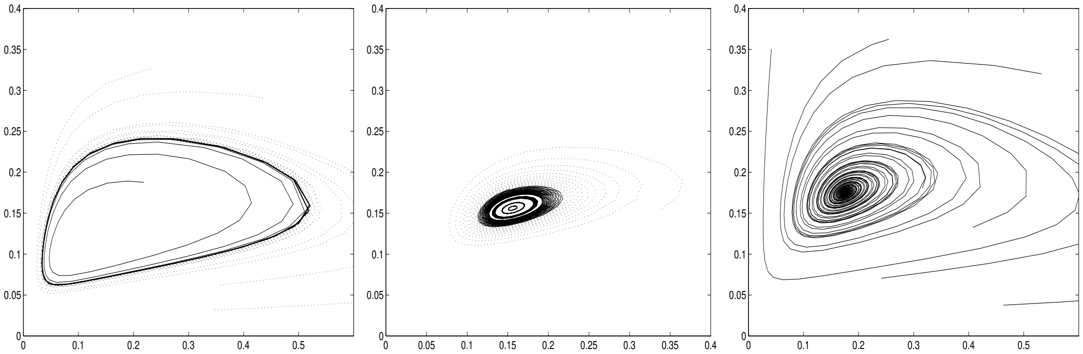

Figure 2: Phase portraits for $\beta = 0.15, 0.2139\ldots$ and 0.25.