75048 Mathematical Modeling 1998

Exercise 6 – Solution

November 12, 1998

Problem 1:

a)

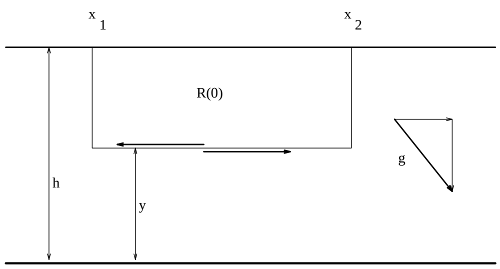

Figure 1: We have drawn the incline horizontally. The direction of $\vec{g}$ is indicated and the x-component is $g \sin \alpha$.

Let the material volume $R(t)$ be such that $R(0)$ becomes as indicated in the figure. If we write $v\vec{\imath}$ for the fluid velocity the change in the x-component of the impulse for the material volume is

$$\frac{d}{dt} \int_{R(t)} \rho v \, dV \bigg|_{t=0} = \int_{R(0)} \rho v_t \, dV + \int_{\partial R(0)} \rho v^2 \vec{\imath} \cdot \vec{n} \, d\sigma = 0.$$

The last equality follows from the fact that $v_t = 0$ (stationary conditions) and the fact that the contributions from the two end pieces in the surface integral cancel since v is only a function of y. The equation for conservation of impulse in the x-direction is thus[1]

$$\frac{d}{dt} \int_{R(t)} \rho v \, dV \bigg|_{t=0} = \int_{R(0)} \rho g \sin \alpha \, dV + \int_{\partial R(0)} \vec{n} \cdot \vec{\vec{T}} \, d\sigma = 0.$$

The pressure part in the stress tensor $\vec{\vec{T}}$ can be ignored as it produces the same integral but with opposite signs over the end pieces $x = x_1$ and $x = x_2$ and on the bottom of the material volume we get $\vec{n} \cdot \vec{\vec{T}} = -\mu(dv/dy)\vec{\imath}$. (This force and its reaction force are indicated with arrows

[1]The significance of the *stress tensor* $\vec{\vec{T}}$ is this: The force acting across an imaginary surface with normal vector n in the fluid is $\vec{n} \cdot \vec{\vec{T}}$ (per unit area). See Lin & Segel section 14.2 (p. 458). In this simple example, we could just have considered these forces directly, without even mentioning the stress tensor.

pointing to the left and right in the figure.) On the top side we assume that $\vec{\vec{T}} = 0$. We can then calculate the integrals above and get

$$0 = \rho g \sin\alpha \cdot (h-y)(x_2 - x_1) - \mu \frac{dv}{dy}(x_2 - x_1)$$

which simplifies to

$$\frac{dv}{dy} = \frac{\rho g \sin\alpha}{\mu}(h-y).$$

Upon integration with respect to y we get

$$v = \frac{\rho g \sin\alpha}{2\mu}(2hy - y^2)$$

plus a constant of integration. This constant has to be zero as $v = 0$ when $y = 0$. This is the non-slip condition and this is also the additional physical constraint it was called for. That the surface tension is zero is another physical condition that could have been mentioned.

The total fluid flux (volume per unit time[2]) past a point on the incline becomes

$$Q(h) = \int_0^h v\, dy = \frac{\rho g \sin\alpha}{2\mu} \int_0^h (2hy - y^2)\, dy = \frac{\rho g \sin\alpha}{3\mu} h^3.$$

b)

The conservation law for fluid becomes

$$\frac{d}{dt}\int_{x_1}^{x_2} h(x,t)\, dx + Q(h(x_2,t)) - Q(h(x_1,t)x_1) = 0.$$

If we write this on differential form and at the same time change variables as indicated in the problem text we arrive at $h^*_{t^*} + Q^*(h^*)_{x^*} = 0$ which upon substitution of $t^* = Tt$, $h^* = Hh$ and $x^* = Xx$ becomes

$$\frac{H}{T}h_t + \frac{H^3}{X}\frac{\rho g \sin\alpha}{3\mu}(h^3)_x = 0$$

which is on the form $h_t + Q(h)_x = 0$, $Q(h) = h^3/3$ provided that

$$\frac{H^2 T}{X}\frac{\rho g \sin\alpha}{3\mu} = \frac{1}{3}.$$

For given H and T we can always set $X = H^2 T \rho g \sin\alpha/\mu$ to obtain this.

c)

The characteristic velocity is $c(h) = Q'(h) = h^2$. In particular the characteristic through points $(0, t_0)$ on the t-axis will have velocity $c(\sqrt{t_0}) = t_0$, and the characteristic therefore has the equation

$$x = t_0(t - t_0).$$

In order to find all characteristics from the t-axis that passes through a given point (x,t) we have to solve the equation above with respect to t_0, and this gives us the desired equation $t_0 = t/2 \pm \sqrt{t^2/4 - x}$.

[2] Or area. We have daringly left out the third coordinate in the entire discussion! Boy, we sure are wild! Multiply with the width of the incline to arrive at volume.

d)

On a characteristics from the x-axis we have $h = 0$ and if (x, t) is a point on the shock then $h(x_+, t) = 0$ while $h(x_-, t) = \sqrt{t_0}$ where $t_0 = t/2 + \sqrt{t^2/4 - x}$. The shock velocity now follows from the Rankine-Hugoniot condition

$$\frac{Q(h(x_+, t)) - Q(h(x_-, t))}{h(x_+, t) - h(x_-, t)} = \frac{0 - t_0^{3/2}}{3(0 - t_0^{1/2})} = \frac{t_0}{3} = \frac{1}{6}(t + \sqrt{t^2 - 4x})$$

and a differential equation for the shock is

$$\dot{s}(t) = \frac{1}{6}(t + \sqrt{t^2 - 4s(t)}).$$

If we substitute $s(t) = \alpha t^2$ and cancel the common factor t, we get $2\alpha = (1/6)(1 + \sqrt{1 - 4\alpha})$ or $(12\alpha - 1)^2 = 1 - 4\alpha$, which has solutions $\alpha = 0$ or $\alpha = 5/36$. We reject $\alpha = 0$ since that is not a solution of the original equation. Moreover, it would have implied that $h = 0$ in the entire first quadrant and this can never satisfy the boundary condition in $x = 0$. So

$$\alpha = \frac{5}{36}$$

leading to $\dot{s}(t) = (5/18)t$, while the two characteristics from $(0, t/2 \pm \sqrt{t^2/4 - s(t)}) = (0, (1/2 \pm (1/3))t)$ have velocities $t/6$ and $5t/6$ (as $c(h(0, t_0)) = t_0$). Because of

$$\frac{t}{6} < \frac{5t}{18} < \frac{5t}{6}$$

for all $t > 0$ this shows that the characteristics from $(0, t/2 + \sqrt{t^2/4 - s(t)})$ intersect the shock from the left as assumed, whereas the one from $(0, t/2 - \sqrt{t^2/4 - s(t)})$ intersects it from the right, after already having intersected it at non-shock point. If this is not clear convince yourself by drawing up some characteristics originating in $(0, t/2 - \sqrt{t^2/4 - s(t)})$ in figure 2.

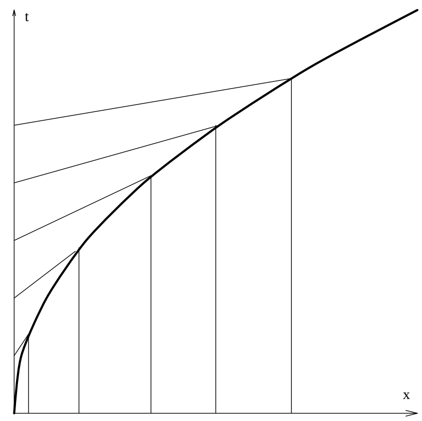

Figure 2: Characteristics and shocks for the solution. If the characteristics from the left are continued through the shock they will eventually hit the shock again from the other side.

Problem 2:

a)

The student flux at a point x^* is $\rho^* v^*$. If we equate the change of students inside $[x_1^*, x_2^*]$ and the difference between in-flux and out-flux we obtain

$$\frac{d}{dt^*}\int_{x_1^*}^{x_2^*} \rho^*(y^*,t^*)\,dy^* = \rho^*(x_1^*,t^*)v^*(x_1^*,t^*) - \rho^*(x^*,t^*)v^*(x^*,t^*). \tag{1}$$

As the running speed v^* decreases linearly from $v_{\max}$ to 0 as ρ^* increases from 0 to $\rho_{\max}$ we get $v^* = v_{\max}(1-\rho^*/\rho_{\max})$. The flux function therefore becomes

$$f^*(\rho^*) = \rho^* v_{\max}\big(1 - \frac{\rho^*}{\rho_{\max}}\big).$$

Upon substitution in (1) we get the conservation law

$$\frac{d}{dt^*}\int_{x_1^*}^{x_2^*} \rho^*(y^*,t^*)\,dy^* + f^*(\rho^*(x_2^*,t^*)) - f^*(\rho^*(x_1^*,t^*)) = 0, \tag{2}$$

or on differential form

$$\rho_t^* + f^*(\rho^*)_x = 0.$$

We introduce the scalings

$$x^* = \frac{L}{2\pi}x, \qquad t^* = \frac{L}{2\pi v_{\max}}t, \qquad v^* = v_{\max}v, \qquad \rho^* = \rho_{\max}\rho,$$

where ρ and v are functions of x and t and arrive at

$$\rho_t + (1-2\rho)\rho_x = 0. \tag{3}$$

Here $0 \le \rho \le 1$. Because of $\rho^*(x^*+L,t^*) = \rho^*(x^*,t^*)$ we get $\rho(x+2\pi,t) = \rho(x,t)$ and ρ is a 2π periodic function in x as claimed.

b)

Note that the condition $0 < \rho_0 \pm \varepsilon < 1$, $\varepsilon > 0$ ensures that $\rho(x,0) > 0$. A characteristic in the xt-plane is curve satisfying

$$x'(t) = 1 - 2\rho(x(t),t). \tag{4}$$

Along a solution curve $x(t)$ of (4) we have $\frac{d}{dt}\rho(x(t),t) = \rho_t + (1-2\rho)\rho_x = 0$, such that ρ is constant on the characteristic. We can write (4) as

$$x'(t) = 1 - 2\rho(x(0),0), \qquad \text{eller} \qquad x(t) = x(0) + \big(1 - 2\rho(x(0),0)\big)t.$$

If we set $x(0) = x_0$ and $\rho(x,0) = \rho_0 + \varepsilon\cos x$ we get that the characteristic through a point x_0 on the x-axis is given by

$$x(t) = x_0 + \big(1 - 2(\rho_0 + \varepsilon\cos x_0)\big)t, \tag{5}$$

where ρ is given by

$$\rho(x,t) = \rho_0 + \varepsilon\cos x_0. \tag{6}$$

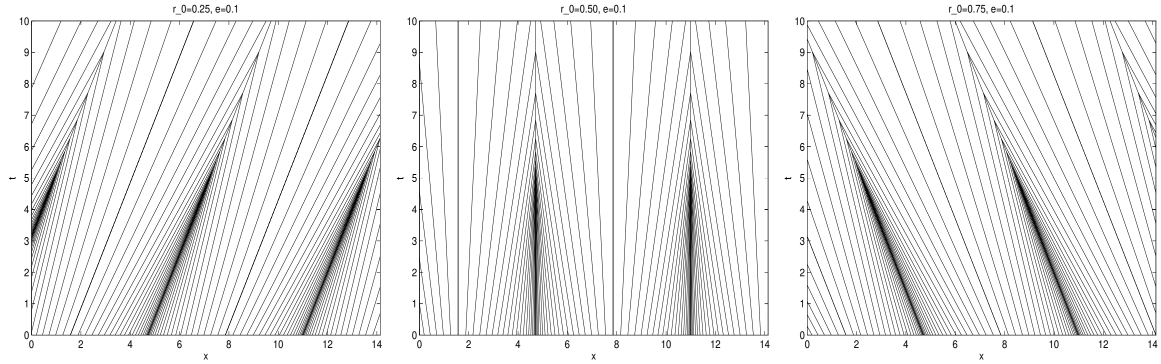

Figure 3: Characteristics for $\rho_0 = 0.25, 0.5$ and 0.75. The shocks first appear at $t = 5$.

c)

The characteristics are given by and figure 3 shows some of the characteristics $\rho_0 = 0.25, 0.5$ and 0.75. Because of $x'(t) > 1 - 2\rho_0$ when $x_0 \in (\frac{1}{2}\pi, \frac{3}{2}\pi)$ and $x'(t) < 1 - 2\rho_0$ when $x_0 \in (\frac{3}{2}\pi, \frac{5}{2}\pi)$, the characteristics from these intervals must intersect. We have $\rho < \rho_0$ for characteristics starting in the first interval, while $\rho > \rho_0$ on the characteristics in the second interval. Consequently, characteristics with different ρ-values will intersect and we get shock solutions.

d)

Consider the point of intersection for the characteristics starting from $x_0 = \frac{3\pi}{2} \pm \theta$ where $0 < \theta < \pi$. As $\cos(\frac{3}{2}\pi \pm \theta) = \pm \sin\theta$ this characteristics are given by

$$x(t) = \frac{3}{2}\pi \pm \theta + \big(1 - 2(\rho_0 \pm \varepsilon \sin\theta)\big)t.$$

At the intersection point of characteristics we have $\theta - 2\varepsilon t \sin\theta = -\theta + 2\varepsilon t \sin\theta$, and the time of intersection is $t_s = \theta/(2\varepsilon \sin\theta)$. When $|\theta| \leq \pi$ this is an increasing function of θ and the closer the characteristics starts relative to $\frac{3}{2}\pi$ the sooner they intersect. Consequently, the time of the shock formation is

$$\min_{\theta\in[-\pi,\pi]} t_s(\theta) = \lim_{\theta\to 0} \frac{\theta}{2\varepsilon \sin\theta} = \frac{1}{2\varepsilon}.$$

We claim that the shock moves along the line ℓ given by

$$x(t) = \frac{3}{2}\pi + (1 - 2\rho_0)t.$$

The collision time $t_s(\theta)$ is an even function of θ, and therefore the characteristic starting in $\frac{3}{2}\pi + \theta$ will hit ℓ at the same time as the characteristic from $\frac{3}{2}\pi - \theta$. Since this time increases with θ no characteristics will intersect outside ℓ, hence the shock follows ℓ.

We arrive at the same conclusion via Rankine-Hugoniot's condition,

$$\frac{dx}{dt} = \frac{\rho_R(1-\rho_R) - \rho_L(1-\rho_L)}{\rho_R - \rho_L} = 1 - \rho_R - \rho_L,$$

where $\rho_L = \rho_0 + \varepsilon \sin\theta$ and $\rho_R = \rho_0 - \varepsilon \sin\theta$. Thus

$$\frac{dx}{dt} = 1 - 2\rho_0.$$

The first shock happens for $t = 1/2\varepsilon$ at $x = \frac{3}{2}\pi + (1-2\rho_0)/2\varepsilon$. By integration it follows that the shock moves along the line ℓ. In the same way all subsequent intersections of characteristics will occur on ℓ. Even though the size of the shock changes the shock condition remains the same because of symmetry and the shock follows ℓ.

Clearly,

$$\rho_L - \rho_R = \rho(\frac{3}{2}\pi - \theta, 0) - \rho(\frac{3}{2}\pi + \theta, 0) \to 0, \qquad \text{as } t_s \to \infty.$$

The shock fades out and the solution approaches the constant solution ρ_0.