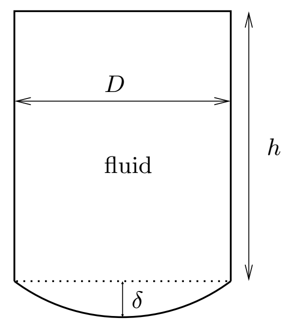
Exercise Set 1

Problem 1



The relevant physical quantities in this problem are the geometry of the vessel (parametrized by d, D and δ), the density ρ and the height h of the liquid which determine the action of the gravity and the modulus of elasticity E which gives a relation between stress and deformation (it relates somehow the pressure and δ).

We sum up in the following table the units of all these quantities

	δ	D	h	d	ρ	g	E
kg	0	0	0	0	1	0	1
m	1	1	1	1	-3	1	-1
s	0	0	0	0	0	-2	-2

The columns given by D, ρ and g are independant. We take these variables as reference variables and we get 7-3=4 independant dimensionless variables:

$$\Pi_1 = \frac{\delta}{D},\ \Pi_2 = \frac{h}{D},\ \Pi_3 = \frac{d}{D},\ \Pi_4 = \frac{E}{D\rho g}$$

The Buckingham's pi theorem tells us that there exists a function Φ such that

$$\Pi_1 = \Phi(\Pi_2, \Pi_3, \Pi_4)$$

i.e.

$$\frac{\delta}{D} = \Phi\left(\frac{h}{D}, \frac{d}{D}, \frac{E}{D\rho g}\right) \tag{1}$$

In fact, the liquid deforms the vessel only through the pressure it exerts at the bottom (which is equal to $\rho g h$). As a consequence $\Pi_2 = \frac{h}{D}$ and $\Pi_4 = \frac{E}{D\rho g}$ are not independant and we combine them to get the pressure term $\rho g h$ explicitly ($\Pi_{new} = \frac{\Pi_4}{\Pi_2} = \frac{E}{\rho g h}$). Hence, we rewrite (1) as

$$\frac{\delta}{D} = \Phi\left(\frac{E}{\rho g h}, \frac{d}{D}\right)$$

We can obtain the same result in a more rigorous way by starting the dimension analysis again. The relevant physical quantities are now δ,D,d,E and $P = \rho g h$.

	δ	D	d	E	P
kg	0	0	0	1	1
m	1	1	1	-1	-1
s	0	0	0	-2	-2

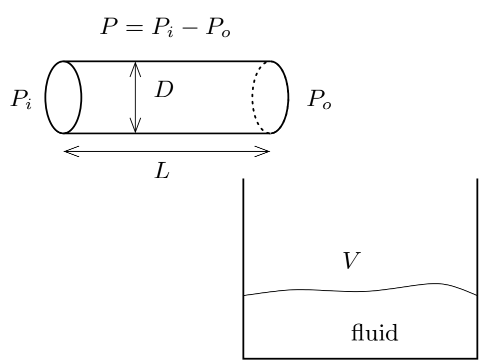
The rank of the system is now 2. We take P and D as reference variables and get 5-2=3 independant variables, namely

$$\Pi_1 = \frac{\delta}{D},\ \Pi_2 = \frac{d}{D},\ \Pi_3 = \frac{E}{P} = \frac{E}{\rho g h}$$

The Buckingham's pi theorem gives us directly that

$$\frac{\delta}{D} = \Phi\left(\frac{E}{\rho g h}, \frac{d}{D}\right)$$

Problem 2



The time required to fill in the vessel depends directly on the flow coming out of the pipe. The flow depends on the diameter of the pipe (D) and on the velocity of the liquid. The striving force in this experiment is the pressure (we assume gravity is not involved) which act through its gradient. Therefore $P = P_i - P_o$ and L must be considered as relevant variables. Roughly speaking, the viscosity μ determines the fluid response to a given excitation and it must be taken into consideration. At first sight it is not clear whether the density ρ plays a role or not (this question arised during the class). ρ is in fact important for modelling turbulent phenomenons but we put it aside for the moment.

We have

	D	L	P	V	t	μ
kg	0	0	1	0	0	1
m	1	1	−1	3	0	−1
s	0	0	−2	0	1	−1

The rank of the system is 3. We take V, t and μ as reference variables. We have 6-3=3 independant variables. Let's take in details the first one.

$$\Pi_1 = \frac{P}{V^{x_1} t^{x_2} \mu^{x_3}}$$

where x_1, x_2, x_3 are solutions of

$$\begin{pmatrix} 0 & 0 & 1 \\ 3 & 0 & -1 \\ 0 & 1 & -1 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 1 \\ -1 \\ -2 \end{pmatrix}$$

We solve this system and get:

$$\Pi_1 = \frac{Pt}{\mu}$$

similarly, we have:

$$\Pi_2 = \frac{D}{V^{1/3}},\ \Pi_3 = \frac{L}{V^{1/3}}$$

The buckingham's pi theorem gives us

$$\frac{Pt}{\mu} = \Phi\left(\frac{D}{V^{1/3}}, \frac{L}{V^{1/3}}\right)$$

Φ depends only on the geometries of the pipe and the vessel which remain unchanged during all the experiments. Therefore

$$\frac{Pt}{\mu} = Constant$$

hence

$$\log(P) = \log(\frac{1}{t}) + \log(\mu) \tag{2}$$

which is exactly what give the graphs! Indeed, in the first graph, we have almost straight lines of slope 1 which only differ by their horizontal position which is determined by μ in (2).

If we take into consideration the density, we obtain an extra dimensionless independant variable. After some computation, we get the following dimensionless variables

$$\Pi_1 = \frac{PV^{2/3}}{\mu^2},\ \Pi_2 = \frac{D}{V^{1/3}},\ \Pi_3 = \frac{L}{V^{1/3}},\ \Pi_4 = \frac{\rho V^{1/3}}{t\mu}$$

and finally, since L, D, V are constant throughout the experiment, we get by the Buckingham's pi theorem

$$\frac{P}{\mu^2} = \Phi\left(\frac{\rho}{\mu}\frac{1}{t}\right)$$

This expression is much less explicit than (2) but it can take care of the fact that we do not have exactly staight lines.