Exercise Set 3

Problem 1

(a) We plug in the expansion of u

$$u = \sum_{n=0}^{\infty} \varepsilon^n u_n$$

in the governing equation

$$u'' + u = 1 + \varepsilon u^2$$

and we get

$$\sum_{n=0}^{\infty} \varepsilon^n u_n'' + \sum_{n=0}^{\infty} \varepsilon^n u_n = 1 + \varepsilon (\sum_{n=0}^{\infty} \varepsilon^n u_n)^2$$

or

$$\sum_{n=0}^{\infty} \varepsilon^n (u_n'' + u_n) = 1 + \varepsilon \sum_{i,j=0}^{\infty} \varepsilon^{i+j} u_i u_j$$

At the order 0, we get

$$u_0'' + u_0 = 1 \tag{1}$$

At the order n, we get

$$u_n'' + u_n = \sum_{\substack{i,j \in \mathbb{N} \\ i+j+1=n}} u_i u_j$$

or

$$u_n'' + u_n = \sum_{i=0}^{n-1} u_i u_{n-1-i} \tag{2}$$

(b) A general solution of (1) is given by

$$u_0(\theta) = 1 + A\cos\theta + B\sin\theta$$

$u(0) = e + 1$ implies that $A = e$ and $u'(0) = 0$ implies that $B = 0$. Hence,

$$u_0(\theta) = 1 + e\cos\theta$$

From (2), we get the equation satisfied by u_1:

$$u_1'' + u_1 = u_0^2$$

or

$$u_1'' + u_1 = (1 + e\cos\theta)^2$$

We expand the the right-hand side and, after using the identity $\cos^2\theta = \frac{1}{2}(\cos 2\theta + 1)$, we get

$$u_1'' + u_1 = (1 + \frac{e^2}{2}) + 2e\cos\theta + \frac{e^2}{2}\cos 2\theta \tag{3}$$

The solution of the homogeneous solution corresponding to (3) is $A\cos\theta + B\sin\theta$. We have to find a particular solution. For the term $\frac{e^2}{2}\cos 2\theta$, a solution of the form $\alpha\frac{e^2}{2}\cos 2\theta$ will do and after some calculation, we get $\alpha = -\frac{1}{3}$. The second term is a bit more tricky since $\cos\theta$ is solution of the homogeneous equation. We want to find a particular solution of

$$v'' + v = \cos\theta \tag{4}$$

We write v as

$$v(\theta) = \alpha(\theta)\cos\theta + \beta(\theta)\sin\theta \tag{5}$$
$$v' = \alpha(\theta)(-\sin\theta) + \beta(\theta)\cos\theta \tag{6}$$

where α, β are unknown functions (such functions allways exist because $\cos\theta$ and $\sin\theta$ are two independant solutions of the homogeneous system).

Then we get

$$0 = \alpha'(\theta)\cos\theta + \beta'(\theta)\sin\theta \tag{7}$$

by differentiating (5) and using (6). We also have

$$\cos\theta = \alpha'(\theta)(-\sin\theta) + \beta'(\theta)\cos\theta \tag{8}$$

because v is solution of (4).

Equations (7) and (8) give us

$$\begin{aligned}\alpha'(\theta) &= -\sin\theta\cos\theta \\ \beta'(\theta) &= \cos^2\theta\end{aligned}$$

that we solve:

$$\begin{aligned}\alpha &= \frac{\cos 2\theta}{4} \\ \beta &= \frac{\sin 2\theta}{4} + \frac{\theta}{2}\end{aligned}$$

hence, we get

$$\begin{aligned}v &= \alpha(\theta)\cos\theta + \beta(\theta)\sin\theta \\ &= \frac{\cos 2\theta}{4}\cos\theta + (\frac{\sin 2\theta}{4} + \frac{\theta}{2})\sin\theta \\ &= \frac{\theta}{2}\sin\theta + \frac{1}{4}\cos\theta \quad \text{(after some computation)}\end{aligned}$$

Since we are only interested in a particular solution, we can drop the $\cos\theta$ term (it is solution of the homogeneous equation) and take $v = \frac{\theta}{2}\sin\theta$. Finally, the solution of (3) is

$$u_1(\theta) = (1 + \frac{e^2}{2}) + e\theta\sin\theta - \frac{e^2}{6}\cos 2\theta + A\cos\theta + B\sin\theta$$

The boundary conditions $u_1(0) = 0$ and $u_1'(0) = 0$ imply that

$$u_1(\theta) = (1 + \frac{e^2}{2}) + e\theta\sin\theta - \frac{e^2}{6}\cos 2\theta - (1 + \frac{e^2}{3})\cos\theta$$

The term $\theta\sin\theta$ is not physical because it grows to infinity. Our approximation is only valid on a small interval when $\theta\sin\theta$ remains of order 1.

(c) We set

$$v(\phi) = u(\theta) \tag{9}$$

where

$$\phi = (1 + \varepsilon h)\theta$$

We differentiate twice (9) and we get

$$(1 + \varepsilon h)^2 v'' = u''$$

We plug in this expression in the governing equation

$$(1 + \varepsilon h)^2 v'' + v = 1 + \varepsilon v^2 \tag{10}$$

We expand v in a power serie of ε up to the order 1

$$v = v_0 + \varepsilon v_1 + o(\varepsilon)$$

and from (10) we get

$$(v_0 + \varepsilon v_1)''(1 + \varepsilon h)^2 + v_0 + \varepsilon v_1 = 1 + \varepsilon(v_0 + \varepsilon v_1)^2 + o(\varepsilon)$$

Equaling the terms of same orders we end up with the following equations that v_0 and v_1 must satisfy

$$v_0'' + v_0 = 1 \tag{11}$$

$$2hv_0'' + v_1'' + v_1 = v_0^2 \tag{12}$$

We have $v_0 = 1 + e\cos\theta$ (v_0 satisfies the same equation with the same boundary conditions as u_0 in the previous question). After some simplification in (12), we get that v_1 satisfies

$$v_1''(\phi) + v_1(\phi) = (1 + \frac{e^2}{2}) + \frac{e^2}{2}\cos 2\phi + 2e(1 + h)\cos\phi \tag{13}$$

The right-hand side is very similar to the one we got for u_1 in the previous question and we use the result we found there to get the general solution of (13)

$$v_1(\phi) = (1 + \frac{e^2}{2}) + e(1+h)\phi \sin\phi - \frac{e^2}{6}\cos 2\phi + A\cos\phi + B\sin\phi$$

We set $h = -1$ so that we get rid of the unphysical term $\phi\sin\phi$. The boundary conditions for v_1 imply that $A = -(1 + \frac{e^2}{3})$ and $B = 0$. We end up with

$$v_1(\phi) = (1 + \frac{e^2}{2}) - \frac{e^2}{6}\cos 2\phi - (1 + \frac{e^2}{3})\cos\phi$$

which is 2π-periodic with respect to ϕ

(d) The system has period 2π with respect to ϕ. $\phi = 2\pi$ when

$$\begin{aligned}\theta &= \frac{2\pi}{1+\varepsilon h} \\ &= 2\pi(1-\varepsilon h) \quad \text{(at first order in } \varepsilon) \\ &= 2\pi + 2\pi\varepsilon\end{aligned}$$

since $h = -1$.

The perihelion (the point where the planet is the closest to the sun) moves forward with an angle $2\pi\varepsilon$ at each rotation.