# Exercise Set 6

## Problem 1

### (a)

The scales can be justified as follows:

| | |
|---|---|
| $x^* = Lx$ | This gives $0 \le x \le 1$ in the problem, and there is no particular reason indicating that $u^*$, $v^*$ and $p^*$ undergo significant changes over smaller distances in the $x^*$-direction. |
| $y^* = \varepsilon L y$ | This gives $0 \le y \le 1$, and $u^*$ changes from $U$ to 0 over the interval $[0, \varepsilon L]$ in the $y^*$-direction. |
| $u^* = Uu$ | Gives roughly $0 \le u \le 1$ in the problem. |
| $v^* = \varepsilon U v$ | Substituting $\lvert\partial_{x^*} u^*\rvert \sim U/L$ and $\lvert\partial_{y^*} v^*\rvert \sim V/(\varepsilon L)$, where $V$ is the scale for $v^*$ into the equation for conservation of mass one obtains $U/L \sim V/(\varepsilon L)$ and $V = \varepsilon U$ as the scaling factor for $v^*$. |

### (b)

With the additional scaling $p^* = Pp$ the system of equations becomes

$$\begin{aligned}
\frac{U^2}{L} u\partial_x u + \frac{U^2}{L} v\partial_y u &= -\frac{P}{L\rho}\partial_x p + \nu\Big(\frac{U}{L^2}\partial_{xx} u + \frac{U}{\varepsilon^2 L^2}\partial_{yy} u\Big)\\
\frac{U^2}{L} u\partial_x v + \frac{U^2}{L} v\partial_y v &= -\frac{P}{\varepsilon^2 L\rho}\partial_y p + \nu\Big(\frac{U}{L^2}\partial_{xx} v + \frac{U}{\varepsilon^2 L^2}\partial_{yy} v\Big)\\
\partial_x u + \partial_y v &= 0.
\end{aligned}$$

The term containing $1/\varepsilon^2$ is clearly the dominating term in the $\nu$–term in the first equation. The assumption that the pressure term containing $p^*_{x^*}$ balances the viscous term leads to $P = \frac{\nu U \rho}{\varepsilon^2 L}$. Substituting this into the equations above leads to

$$\varepsilon\,\mathrm{Re}(u\partial_x u + v\partial_y u) = -\partial_x p + \varepsilon^2\partial_{xx} u + \partial_{yy} u, \tag{1}$$

$$\varepsilon^3\mathrm{Re}(u\partial_x v + v\partial_y v) = -\partial_y p + \varepsilon^4\partial_{xx} v + \varepsilon^2\partial_{yy} v, \tag{2}$$

$$\partial_x u + \partial_y v = 0. \tag{3}$$

### (c)

If we try the value $\nu \approx 1\,\mathrm{centiStokes} = 10^{-2}\,\mathrm{cm}^2/\mathrm{s} = 10^{-6}\,\mathrm{m}^2/\mathrm{s}$ together with the other given values we find $\mathrm{Re} \approx 100$. Only if the viscosity is an order of magnitude smaller, do we approach the critical limit $\mathrm{Re} \approx 1000$ – so it appears safe to assume that turbulence will not arise.

### (d)

By setting $\varepsilon$ to 0 in (1)–(3) one obtains

$$\partial_x p = \partial_{yy} u, \quad \partial_y p = 0, \quad \text{and} \quad \partial_x u + \partial_y v = 0. \tag{4}$$

Since $p$ does not depend on $y$, the first equation gives

$$u = \frac{1}{2} y^2 \partial_x p + Ay + B.$$

The boundary conditions $u(x,0)=1$ and $u(x,h(x))=0$ give $B=1$ and $A=-\frac{1}{2}h(x)\partial_x p - 1/h(x)$ and thus

$$u = -\frac{1}{2}y(h-y)\partial_x p + 1 - \frac{y}{h}. \tag{5}$$

Integrating the third equation in (4) with respect to $y$ and using the boundary conditions $v(x,0)=v(x,h(x))=0$ gives

$$0 = \int_0^{h(x)} (\partial_x u + \partial_y v)dy = \int_0^{h(x)} \partial_x u \, dy + v(x,h(x)) - v(x,0) = \int_0^{h(x)} \partial_x u \, dy,$$

and substituting $\partial_x u$ from (5) gives

$$\begin{aligned} 0 &= \int_0^{h(x)} \partial_x\big[-\tfrac{1}{2}y(h-y)\partial_x p + 1 - \frac{y}{h}\big]\, dy \\ &= \partial_x\Big(\int_0^{h(x)} \big[-\frac{1}{2}y(h-y)\partial_x p + 1 - \frac{y}{h}\big]\, dy\Big) \\ &= \partial_x\Big(-\frac{1}{12}(h^3\partial_x p) + \frac{1}{2}h\Big). \end{aligned}$$

(When we moved the derivative outside the integral sign we should have subtracted a term due to the $x$ dependency in the upper limit of the integral, but this term is zero because the integrand is zero at $y=h$.) Integration gives $\frac{1}{6}h^3\partial_x p = h - \overline{h}$, where $\overline{h}$ is a constant of integration. This can be written $\partial_x p = 6(h-\overline{h})/h^3$, which is Reynold's equation.

Finally, by setting $h = 1-\alpha x$ one obtains

$$\partial_x p = 6\left(\frac{1}{(1-\alpha x)^2} - \frac{\overline{h}}{(1-\alpha x)^3}\right).$$

Another integration using $p(0)=0$ gives

$$\begin{aligned} p(x) &= 6\left[\frac{1}{\alpha(1-\alpha x)} - \frac{\overline{h}}{2\alpha(1-\alpha x)^2}\right]_0^x \\ &= \frac{6}{\alpha(1-\alpha x)^2}\Big[(1-\alpha x) - \frac{\overline{h}}{2} - (1-\alpha x)^2 + \frac{\overline{h}}{2}(1-\alpha x)^2\Big] \\ &= \frac{6}{\alpha(1-\alpha x)^2}\Big(-\alpha x + \alpha^2 x^2 - \frac{\overline{h}}{2}(\alpha^2 x^2 - 2\alpha x)\Big). \end{aligned}$$

Using $p(1)=0$ gives $0 = -\alpha + \alpha^2 - \frac{\overline{h}}{2}(\alpha^2 - 2\alpha)$ and therefore $\overline{h} = 2(\alpha-1)/(\alpha-2)$.

(Note that $\overline{h} = \frac{2}{1/(1-\alpha)+1/1}$. Hence $\overline{h}$ is the harmonic mean of 1 and $1-\alpha$ (the height at $x=0$ and $x=1$ respectively), and thus $1-\alpha < \overline{h} < 1$.)

The maximum pressure is attained at $h=\overline{h}$ since $\frac{\partial p}{\partial x}(h=\overline{h})=0$. We find

$$p_{\max} = \frac{3}{2}$$

Hence,

$$p^*_{\max} = \frac{3U\nu\rho}{2L\varepsilon^2} = 30 \text{ hPa}$$