Exercise Set 7

Problem 1 The system is

$$\dot{x} = x + 2y - x(x^4 + y^4) \tag{1}$$
$$\dot{y} = -2x + y - y(x^4 + y^4) \tag{2}$$

The critical points are solutions of

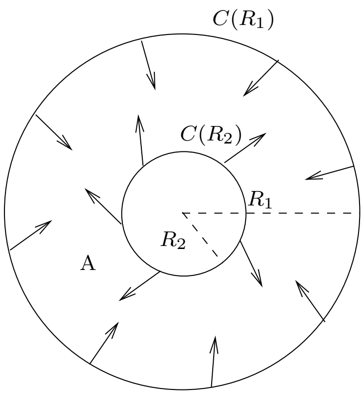
$$x + 2y - x(x^4 + y^4) = 0$$
$$-2x + y - y(x^4 + y^4) = 0$$

We multiply the first equation by y, the other by x and take the difference. We get

$$x^2 + y^2 = 0$$

Hence, $(0, 0)$ is the unique critical point of the system. One can see that it is an unstable focus by looking at the eigenvalues of the linearized system.

Consider the closed annulus A delimited by the two circles $C(R_1)$ and $C(R_2)$ of center 0 and radius R_1 and R_2 respectively. A is a closed bounded region which contains no critical point. We are going to prove that, provided R_1 is big enough and R_2 small enough, any path that lies in A at some time t_0 remains in A for all $t > t_0$. Then the Poincaré-Bendixon theorem says that there exists a closed path.



The trajectory cannot leave the annulus A
(The arrows are just indicative)

We multiply (1) by x and (2) by y, add the two resulting equations and get

$$\dot{x}x + \dot{y}y = x^2 + y^2 - (x^2 + y^2)(x^4 + y^4)$$

or

$$\frac{d}{dt}\|r(t)\|^2 = 2(x^2+y^2)(1-x^4-y^4) \tag{3}$$

where r denotes the vector $(x,y)^t$ and $\|r\| = (x^2+y^2)^{1/2}$ is the standard euclidian norm.

When x^2+y^2 tends to ∞, the right-hand side in (3) tends to $-\infty$. Hence, we can choose R_1 so that

$$\frac{d}{dt}\|r\|^2 < -1 \tag{4}$$

for any t such that $\|r(t)\| = R_1$ (which means $r(t) \in C(R_1)$). Then it is pretty clear that a trajectory cannot leave the annulus through a point of $C(R_1)$ because the flow pushes any point on $C(R_1)$ back towards the center. If one wants to give a detailed proof of this statement, one can proceed as follows.

Consider now a path $r(t)$ which for some t_0 lies in A. Assume that it does not remain in A. Then we can define the time $\bar{t}$ when $r(t)$ first leaves A:

$$\bar{t} = \sup\{t \geq t_0 \mid r(\tilde{t}) \in A,\ \forall \tilde{t} \leq t\}$$

We have $r(\bar{t}) \in \partial A$. We first consider the case when $r(t)$ leaves A by a point of $C(R_1) : r(\bar{t}) \in C(R_1)$. By definition of $\bar{t}$, there exists a sequence t_n converging to $\bar{t}$ such that $r(t_n) \in A^c$. In the case we are considering where $r(\bar{t}) \in C(R_1)$ we necessarily have

$$\|r(t_n)\| > R_1$$

But

$$\|r(t_n)\|^2 > R_1^2 \text{ and } \|r(\bar{t})\|^2 = R_1^2$$

implies that

$$\frac{d}{dt}\|r(t)\|^2_{|t=\bar{t}} \geq 0 \tag{5}$$

which contradicts (4)

In a similar way, one can prove that $r(t)$ never leaves A through the circle $C(R_2)$. In this case we have to take R_2 small enough so that for some $\varepsilon > 0$

$$\frac{d}{dt}\|r(t)\|^2 > \varepsilon$$

for any t such that $\|r(t)\| = R_2$. Hence we have proved that $r(t)$ remains in A for $t \geq t_0$.

Problem 2

(a)

$$\begin{aligned}\dot{x} &= y\\ \dot{y} &= (x^2+1)y - x^5\end{aligned}$$

We have

$$P_x + Q_y = x^2 + 1 > 0$$

Therefore the system does not admit a periodic solution.

(b)

$$\dot{x} = y$$
$$\dot{y} = y^2 + x^2 + 1$$

Since $\dot{y}$ is strictly positive, y is strictly increasing and we have

$$y(0) < y(T) \quad \forall T > 0$$

Hence the system cannot admit a periodic solution.

(c)

$$\dot{x} = y$$
$$\dot{y} = 3x^2 - y - y^5$$

$$P_x + Q_y = -1 - y^4 < 0$$

and the system does not admit a periodic solution.

Problem 3

(a) The system has three equilibrium points :

$$P = 0, \; P = m, \; P = M$$

If we set

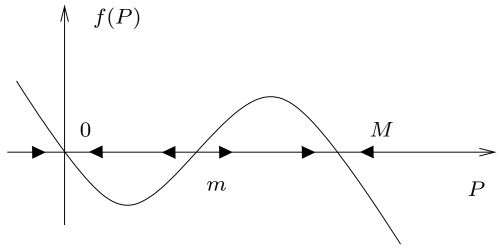
$$f(P) = kP(1 - \frac{P}{M})(\frac{P}{m} - 1)$$

the system writes

$$\dot{P} = f(P)$$

$f_P(0) = -k < 0$ and $f_P(M) = -k(\frac{M}{m} - 1) < 0$ imply that 0 and M are stable equilibrium points while m is an unstable equilibrium point because $f_P(m) = k(1 - \frac{m}{M}) > 0$.

If the initial number of moose $P(0)$ lies between 0 and m then the population dies out. $P(t)$ converges to the equilibrium point 0. If $P(0)$ is bigger than m then the population stabilizes around M.



(b) The equilibrium points of the system are solutions of

$$P(1-P) - J = 0$$
$$-\frac{1}{2}J + JP = 0$$

which gives three equilibrium points

$$(P, J) = (0,0),\ (0,1),\ (\frac{1}{2}, \frac{1}{4})$$

At $(P, J) = (0, 0)$, a linearization of the system gives the matrix

$$\begin{pmatrix} 1 & -1 \\ 0 & -\frac{1}{2} \end{pmatrix}$$

whose eigenvalues are $-\frac{1}{2}$ and 1. $(2,1)^t$ and $(1,0)^t$ are two corresponding eigenvectors. $(0,0)$ is a saddle.

At $(P, J) = (\frac{1}{2}, \frac{1}{4})$, the linearized system gives rise to the matrix

$$\begin{pmatrix} 0 & -1 \\ \frac{1}{4} & 0 \end{pmatrix}$$

whose eigenvalues are purely imaginary. Hence $(\frac{1}{2}, \frac{1}{4})$ is a center for the linearized system, but not necessarily for the nonlinear system.

At $(1, 0)$, the linearized system is given by the matrix

$$\begin{pmatrix} -1 & -1 \\ 0 & \frac{1}{2} \end{pmatrix}$$

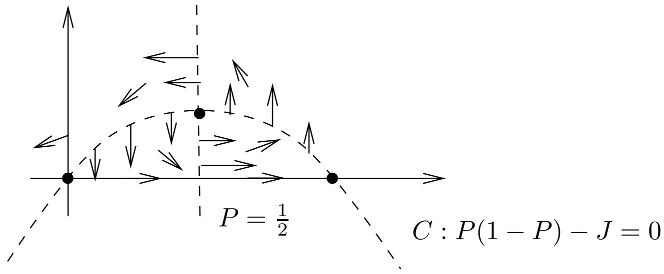
and the eigenvalues are -1 and $\frac{1}{2}$. $(1,0)$ is a saddle.

We now plot the phase plane diagram of the system. $\dot{P}$ vanishes on the curve C:

$$J = P(1-P)$$

while $\dot{J}$ vanishes when

$$P = \frac{1}{2} \text{ or } J = 0$$



We now claim that there exists a family of closed paths which circle around the equilibrium point $(\frac{1}{2}, \frac{1}{4})$. In order to prove that we consider the trajectory $(P(t), J(t))$ solution of the system for the initial condition

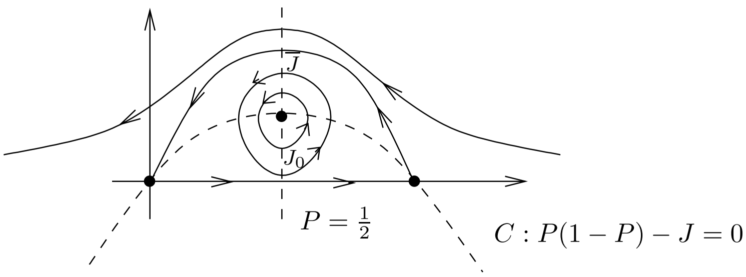
$$P(0) = \frac{1}{2},\ J(0) = J_0$$

with $J_0 \in (0, \frac{1}{4})$.

One can prove that $(P(t), J(t))$ successively hits C and the line $P = \frac{1}{2}$. So there exists $\bar{t}$ such that

$$P(\bar{t}) = \frac{1}{2}$$

We denote $\overline{J}$ the value of J at $\bar{t}$.



The system is invariant under the transformation $P \curvearrowright 1-P$, $t \curvearrowright -t$ which means that

$$\tilde{P}(t) = 1 - P(-t)$$
$$\tilde{J}(t) = J(-t)$$

is also solution of the system. The system is also invariant under time translation (one can shift the time origin). Therefore, we can reset $\tilde{P}$, $\tilde{J}$ as

$$\tilde{P}(t) = 1 - P(-t + 2\bar{t})$$
$$\tilde{J}(t) = J(-t + 2\bar{t})$$

and $\tilde{P}$, $\tilde{J}$ are still solutions of the system.

However, since we have,

$$\tilde{P}(\bar{t}) = \frac{1}{2}, \tilde{J}(\bar{t}) = \overline{J}$$

$(\tilde{P}, \tilde{J})$ and (P, J) are equal at $\bar{t}$. The fact that the solution of the system is unique when the initial condition are the same that

$$\tilde{P} = P \text{ and } \tilde{J} = J$$

Taking $t = 0$ implies

$$\begin{aligned} P(0) &= 1 - P(2\bar{t}) \\ J(2\bar{t}) &= J(0) \end{aligned}$$

hence

$$\begin{aligned} P(2\bar{t}) &= P(0) \\ J(2\bar{t}) &= J(0) \end{aligned}$$

This implies that the solution is periodic because $(P(t + 2\bar{t}), J(t + 2\bar{t}))$ and $(P(t), J(t))$ are now two solutions of the system for the same initial condition. By unicity of the solution, we must have

$$\begin{aligned} P(t + 2\bar{t}) &= P(t) \\ J(t + 2\bar{t}) &= J(t) \end{aligned}$$