Exercise Set 10

a)

The center of the svarm is the point x_c satisfying

$$\int_{-\infty}^{x_c} \rho(s^*, t^*)\, ds^* = \int_{x_c}^{\infty} \rho(s^*, t^*)\, ds^*$$

$w(x^*, t^*)$ can then be rewritten as

$$w(x^*, t^*) = -K \int_{x_c}^{x^*} \rho(s^*, t^*)\, ds^*$$

If an insect stands at the left of the center ($x^* < x_c$) then it gets a positive speed, if it stands at the right, it gets a negative speed. In both cases the insect will fly in the direction of the center. In addition, the further the insect stands from the center, the bigger will be its speed.

We takeaking into account the diffusion effect and get the following conservation law:

$$\frac{d}{dt^*} \int_{-\infty}^{x^*(t)} \rho(s^*, t^*)\, ds^* - \sigma \frac{\partial \rho}{\partial x^*}(x^*(t), t) = 0$$

which implies

$$\int_{-\infty}^{x^*} \frac{\partial \rho}{\partial t^*}(s^*, t^*)\, ds^* + w(x^*, t)\rho(x^*, t^*) - \sigma \frac{\partial \rho}{\partial x^*}(x^*, t) = 0 \tag{1}$$

b)

w can be rewritten in term of the new variable v^*:

$$w(x^*, t^*) = K(M - 2v^*)$$

Since we have

$$\int_{-\infty}^{x^*} \frac{\partial \rho}{\partial t^*}(s^*, t^*)\, ds^* = \frac{\partial v^*}{\partial t^*} \qquad \rho(x^*, t^*) = \frac{\partial v^*}{\partial x^*} \qquad \frac{\partial \rho}{\partial x^*} = \frac{\partial^2 v^*}{\partial x^{*2}}$$

The conservation law (1) becomes

$$\frac{\partial v^*}{\partial t^*} = \varepsilon \frac{\partial^2 v^*}{\partial x^{*2}} - K(M - 2v^*) \frac{\partial v^*}{\partial x^*} \tag{2}$$

c)

Natural scalings for x^* and v^* are

$$x = \frac{x^*}{L} \quad v = \frac{v^*}{M}$$

For the moment, let's consider the unknown scaling in time

$$t = \frac{t^*}{T}$$

We can write equation (2) in a dimensionless form

$$\frac{\partial v}{\partial t} = \frac{T\sigma}{L^2}\frac{\partial^2 v}{\partial x^2} - \frac{KMT}{L}(1-2v)\frac{\partial v}{\partial x} \tag{3}$$

T_D and T_K as defined in the text respectively appear as the characteristic diffusion and convection times (for example, if $T = \mathcal{O}(T_D)$ the evolution term $\frac{\partial v}{\partial t}$ and the diffusion are of the same order).

We choose $T = T_K$. Equation (3) becomes

$$\frac{\partial v}{\partial t} = \sigma\frac{\partial^2 v}{\partial x^2} - (1-2v)\frac{\partial v}{\partial x} \tag{4}$$

with

$$\varepsilon = \frac{T_K}{T_D}$$

which is the ratio between the diffusion and convection time.

d)

We neglect the diffusion and consider the equation

$$v_t + (1-2v)v_x = 0 \tag{5}$$

The characteristics' equations are given by

$$x = (1 - 2v(\xi, 0))t + \xi$$

The initial value of v, $v(x, 0)$, is easily computed from $\rho^*(x^*, 0)$. We have

$$v(x,0) = \begin{cases} 0, & x \le -1 \\ (x+1)/2, & |x| \le 1 \\ 1, & x \ge 1 \end{cases}$$

The characteristics are

$$\begin{array}{rcll} x & = & t + \xi & \text{if } \xi \le -1 \\ x & = & -\xi t + \xi & \text{if } |\xi| \le 1 \\ x & = & -t + \xi & \text{if } \xi \ge 1 \end{array}$$

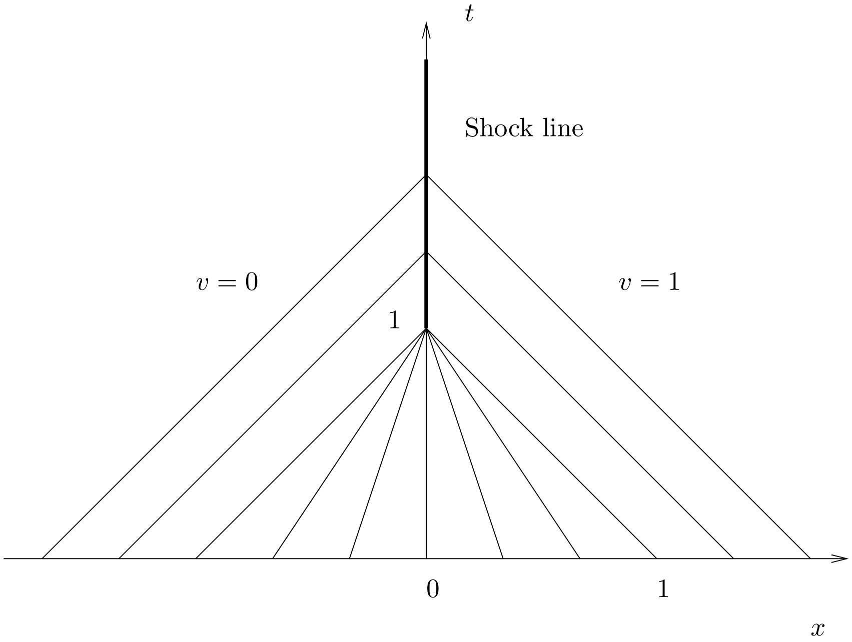
When $|\xi| \leq 1$, the characteristics all meet at $x = 0$ at time $t = 1$. A shock forms there. On the left of the shock $v_l = 0$, on the right $v_r = 1$. The Rankine-Hugoniot condition gives us the speed s_c of the shock:

$$s_p = \frac{(v_l - v_l^2) - (v_r - v_r^2)}{v_l - v_r}$$

Hence

$$s_p = 0$$

On the line $x = 0$, for $t > 1$, the Rankine-Hugoniot conditions are therefore satisfied. We have a shock that remains at $x = 0$.



For $t < 1$, the solution is given by the characteristics

$$v(x,t) = \begin{cases} 0, & x \leq -(1-t) \\ \frac{1}{2}\left(\frac{x}{1-t} + 1\right), & |x| \leq 1-t \\ 1, & x \geq 1-t \end{cases}$$

for $t \geq 1$, only a single shock remains

$$v(x,t) = \begin{cases} 0, & x \leq 0 \\ 1, & x > 0 \end{cases}$$

We reintroduce the dimensions in these equations. ρ^* is then given by $\frac{\partial v^*}{\partial x^*}$.

For $t^* \leq \frac{L}{MK}$,

$$\rho^*(x^*, t^*) = \begin{cases} 0, & x^* \leq -L + MKt^* \\ \frac{M}{2(L-MKt^*)}, & |x^*| \leq L - MKt^* \\ 1, & x \geq L - MKt^* \end{cases}$$

When $t > 1$,

$$\rho^*(x^*, t^*) = \delta(x^*)$$

ρ^* is equal to the Dirac function.

The Dirac function is not physical. If we take into account the diffusion, the shock disappear and ρ^* is a smooth function.

e)

We want to look at solutions when t tends to ∞. In this case, the problem certainly reach an equilibrium and the two leading terms in the equation are the diffusion and convection terms while $c_t \ll 1$. We set

$$x^* = L'x \qquad t^* = Ttv^* \qquad = Mv \tag{6}$$

and get

$$\frac{1}{T}v_t = \frac{\sigma}{L'^2}v_{xx} - \frac{KM}{L'}(1-2v)v_x$$

We choose L' so that the diffusion and convection terms are of the same order:

$$L' = \frac{\sigma}{KM}$$

The evolution term can be neglicted if

$$T \gg \frac{\sigma}{K^2M^2} = T_K\frac{T_D}{T_K} = T_K$$

(with our new scaling L', the diffusion time equals the convection time).

We get

$$v_{xx} = (1-2v)v_x \tag{7}$$

From the ρ^* we are given, we get after integration

$$v = \tilde{A}(\tanh(\tilde{B}x) + 1) \tag{8}$$

for some constants $\tilde{A}$ and $\tilde{B}$.

Since

$$\lim_{x\to\infty} v(x,t) = 1$$

(the total number of insect is constant and equal to M), we have

$$\tilde{A} = \frac{1}{2}$$

Plugging (8) into (7), we get

$$-\tanh(\tilde{B}x)(1 - \tanh^2(\tilde{B}x))\tilde{B}^2 = -\frac{1}{2}\tanh(\tilde{B}x)(1 - \tanh^2(\tilde{B}x))\tilde{B}$$

Hence

$$\tilde{B} = \frac{1}{2}$$

and

$$v = \frac{1}{2}\big(\tanh(x/2) + 1\big)$$

which gives

$$\rho = \frac{1}{4\cosh^2(x/2)}$$

If we reintroduce the dimensional quantities, we get

$$\rho(x^*) = \frac{A}{\cosh^2(Bx^*)}$$

where $A = \frac{KM^2}{4\sigma}$ and $B = \frac{KM}{2\sigma}$.