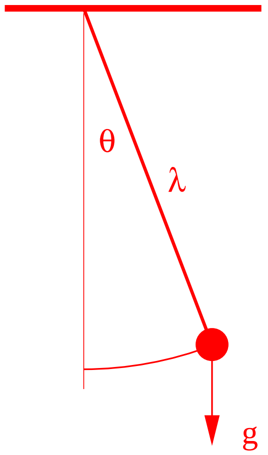
Buckingham's magical pi theorem

and the Lie symmetries of nature

Harald Hanche-Olsen

A simple example



The period t depends on λ, g, $\theta_{\max}$. But how?

Consider the units:
$[t] =$ T; $[\lambda] =$ L; $[g] =$ LT^{-2}; $[\theta_{\max}] = 1$.

The only dimensionally correct combination:

$$t = f(\theta_{\max})\sqrt{\frac{\lambda}{g}}$$

Buckingham's pi theorem formalizes this procedure.

The pendulum equation

$$\lambda\ddot{\theta} + g\sin\theta = 0$$

Motivated by earlier analysis, introduce dimensionless time t^* by

$$t = \sqrt{\frac{\lambda}{g}}t^*$$

and get the dimensionless form of the equation:

$$\ddot{\theta} + \sin\theta = 0$$

Edgar Buckingham (1867–1940)

Educated at Harvard and Leipzig, worked at the (US) National Bureau of Standards 1905–1937. (Soil physics, gas properties, acoustics, fluid mechanics, blackbody radiation.)

On Physically Similar Systems: Illustrations of the Use of Dimensional Equations. *Physical Review* **4**, 345–376 (1914).

ON PHYSICALLY SIMILAR SYSTEMS; ILLUSTRATIONS OF THE USE OF DIMENSIONAL EQUATIONS.

By E. Buckingham.

1. *The Most General Form of Physical Equations.*—Let it be required to describe by an equation, a relation which subsists among a number of physical quantities of n different kinds. If several quantities of any one kind are involved in the relation, let them be specified by the value of any one and the ratios of the others to this one. The equation will then contain n symbols $Q_1 \cdots Q_n$, one for each kind of quantity, and also, in general, a number of ratios r', r'', etc., so that it may be written

$$f(Q_1, Q_2, \cdots Q_n, r', r'', \cdots) = 0. \qquad (1)$$

Let us suppose, for the present only, that the ratios r do not vary during the phenomenon described by the equation; for example, if the

The Framework

Physical quantities: $W_1, W_2, \ldots, W_n$
Expressed in fundamental units $\mathsf{L}_1, \mathsf{L}_2, \ldots, \mathsf{L}_m$

$$W_j = W_j^{\#}[W_j]; \qquad W_j^{\#} \in \mathbb{R}, \quad [W_j] = \prod_{i=1}^{m} \mathsf{L}_i^{a_{ij}}$$

Combinations of physical quantities:

$$\mathbf{W}^{\mathbf{x}} = \prod_{j=1}^{n} W_j^{x_j}, \qquad \mathbf{x} \in \mathbb{R}^n$$

What are the units of $\mathbf{W}^{\mathbf{x}}$?

How units combine

$$[\mathbf{W}^{\mathbf{x}}] = \prod_{j=1}^{n}\prod_{i=1}^{m} \mathsf{L}_i^{a_{ij}x_j} = \prod_{i=1}^{m}\prod_{j=1}^{n} \mathsf{L}_i^{a_{ij}x_j} = \prod_{i=1}^{m} \mathsf{L}_i^{\sum_{j=1}^{n} a_{ij}x_j}$$

Introduce the dimension vectors: $\{W_j\} = (a_{1j}, \ldots, a_{mj})^{\mathsf{T}}$ which form the columns of the dimension matrix $\mathbf{A}$. Formally:

$$[\mathbf{W}^{\mathbf{x}}] = \mathsf{L}^{\mathbf{A}\mathbf{x}}$$

i.e., by linear algebra!

The combination $[\mathbf{W}^{\mathbf{x}}]$ is dimensionless iff $\mathbf{A}\mathbf{x} = 0$.

Dimensionless combinations

$$[\mathbf{W}^{\mathbf{x}}] = \mathsf{L}^{\mathbf{A}\mathbf{x}}$$

Combinations $\mathbf{W}^{\mathbf{z}_\nu}$ are called independent if the vectors $\mathbf{z}_\nu$ are linearly independent.

We can create a
maximal independent set of dimensionless combinations

$$\Pi_\nu = \mathbf{W}^{\mathbf{z}_\nu}, \quad \nu = 1, \ldots, k$$

by letting $\mathbf{z}_1, \ldots, \mathbf{z}_k$ be a basis for $\ker \mathbf{A}$.

New variables

By expanding to a basis for $\mathbb{R}^n$, we get an independent set of combinations:

$$\Pi_1, \ldots, \Pi_k, X_1, \ldots, X_{n-k}$$

Each of the original variables W_j is a combination of these!

Hence we may, and shall, rewrite any problem in terms of the new variables.

Some physics

A general physical law:

$$F(W_1, \dots, W_n) = 0$$

Which means:

$$F^{\#}(W_1^{\#}, \dots, W_n^{\#}) = 0$$

But most importantly: The form of this equation is invariant with respect to a change of units.

Since F is a result of computing with $(W_1, \dots, W_n)$, the units of F must be the units of a combination of $(W_1, \dots, W_n)$. Thus we may assume WOLOG that F is dimensionless.

Independence of units

Introduce new units $\tilde{\mathsf{L}}_i$ by $\mathsf{L}_i = e^{c_i}\tilde{\mathsf{L}}_i$

$$W_j = \widetilde{W}_j^{\#} \prod_{i=1}^{m} \tilde{\mathsf{L}}_i^{a_{ij}} = W_j^{\#} \prod_{i=1}^{m} \mathsf{L}_i^{a_{ij}} = W_j^{\#} \exp\Big(\sum_{i=1}^{m} c_i a_{ij}\Big) \prod_{i=1}^{m} \tilde{\mathsf{L}}_i^{a_{ij}}$$

and so

$$\widetilde{W}_j^{\#} = e^{\mathbf{c}\{W_j\}} W_j^{\#}$$

The invariance under change of units thus means

$$F(W_1, \dots, W_n) = F(e^{\mathbf{c}\{W_1\}} W_1, \dots, e^{\mathbf{c}\{W_n\}} W_n) \qquad (\mathbf{c} \in \mathbb{R}^m)$$

i.e., the equation is invariant under the action of an m-parameter group of symmetries.

Buckingham's theorem

Any dimensionally correct relationship involving physical quantities can be expressed in terms of a maximal set of dimensionless combinations of the given quantities:

$$\Phi(\Pi_1, \ldots, \Pi_k) = 0.$$

The proof

Rewrite the physical law in terms of these variables:

$$\Phi(\Pi_1, \ldots, \Pi_k, X_1, \ldots, X_{n-k}) = 0$$

The invariance:

$$\Phi(\Pi_1, \ldots, \Pi_k, X_1, \ldots, X_{n-k}) = \Phi(\Pi_1, \ldots, \Pi_k, e^{\mathbf{c}\{X_1\}} X_1, \ldots, e^{\mathbf{c}\{X_{n-k}\}} X_{n-k})$$

Linear algebra tells us that the vectors

$$(\mathbf{c}\{X_1\}, \ldots, \mathbf{c}\{X_{n-k}\}), \qquad \mathbf{c} \in \mathbb{R}^m$$

fill all of $\mathbb{R}^{n-k}$, and it follows that
Φ depends only on the dimensionless combinations $(\Pi_1, \ldots, \Pi_k)$.

Example: Water waves

The speed v of a (not too small) water wave in deep water depends on the wave length λ, acceleration of gravity g, and (perhaps) the density ρ of water.

$$[\lambda] = \mathsf{L}, \quad [v] = \mathsf{LT}^{-1}, \quad [g] = \mathsf{LT}^{-2}, \quad [\rho] = \mathsf{ML}^{-3}.$$

Since only ρ contains M, no dimensionless combination can involve ρ. We find one dimensionless combination:

$$\Pi = v^2 g^{-1} \lambda^{-1}.$$

Thus

$$v \propto \sqrt{g\lambda}.$$

Example: Nuclear explosions

Sir Geoffrey Taylor: The formation of a blast wave by a very intense explosion. I & II. *Proc. Royal Soc.* (*London*) **201A**, 159–186 (1950).

Radius r of the fireball is a function of time t, initial energy E, and initial densityρ_0 of air.

$$[r] = \mathsf{L}, \quad [t] = \mathsf{T}, \quad [E] = \mathsf{ML}^2\mathsf{T}^{-2}, \quad [\rho_0] = \mathsf{ML}^{-3}$$

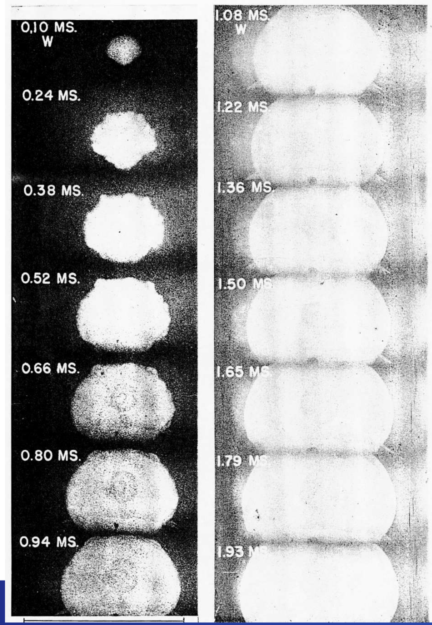
Just one dimensionless combination:

$$\Pi = r^5 t^{-2} \rho_0 E^{-1}$$

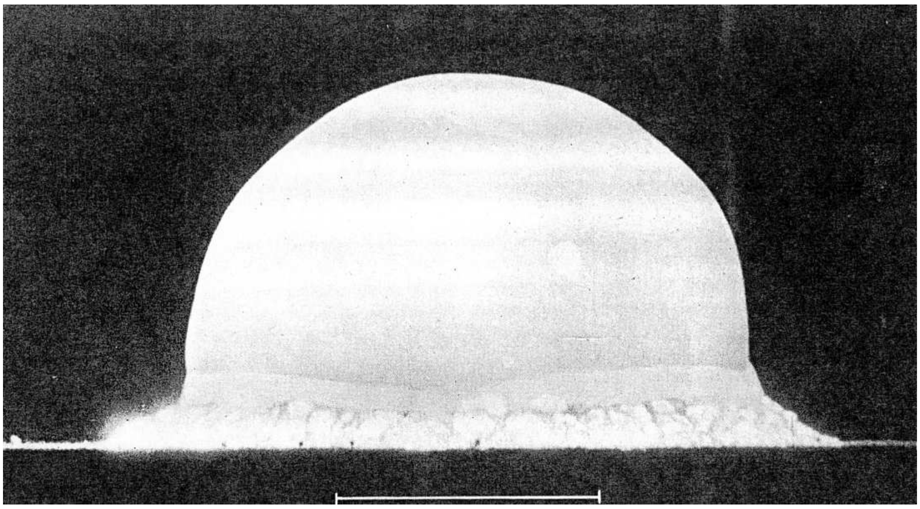
leading to

$$r \propto \left(\frac{E}{\rho_0}\right)^{1/5} t^{2/5}.$$

Nuclear explosion: First 2 ms



Nuclear explosion at 15 ms



The bar is 100 m long

Nuclear explosion at 127 ms

Nuclear yield

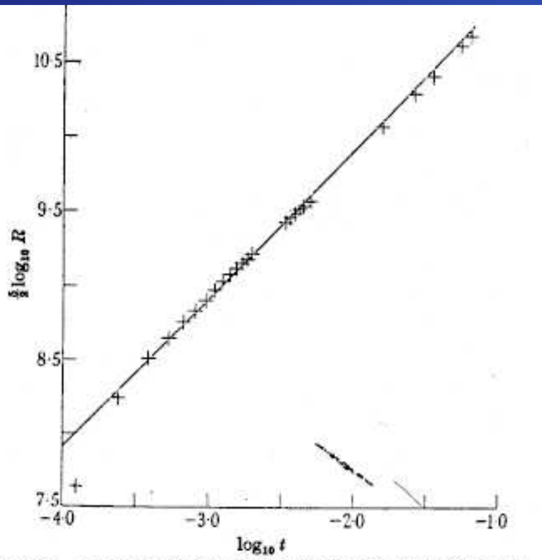


FIGURE 1. Logarithmic plot showing that $R^{\frac{5}{2}}$ is proportional to t.

Example: Fluid flow in pipes

Pressure drop per unit length dP/dx as a function of density ρ, viscosity μ, average flow velocity U, pipe diameter D:
Two dimensionless combinations:

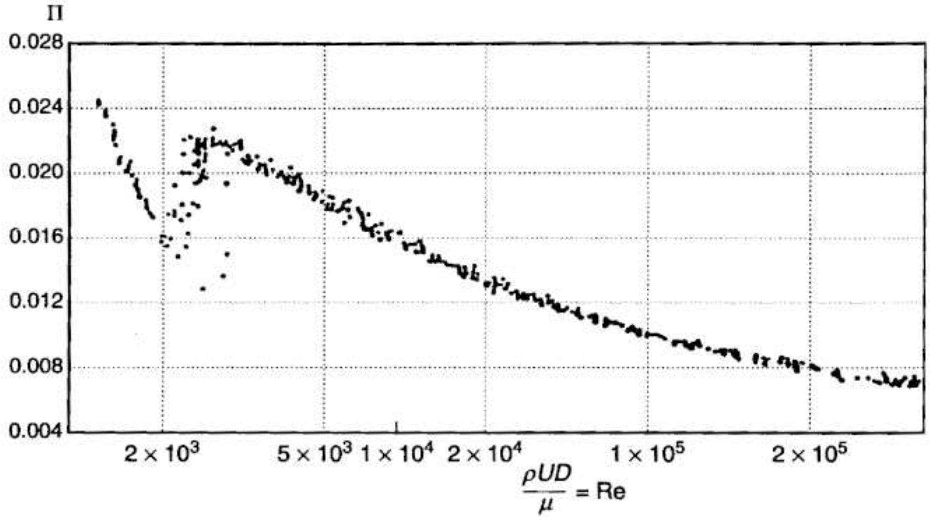
$$\Pi = \frac{dP/dx \cdot D}{U^2 \rho}, \quad \mathrm{Re} = \frac{\rho U D}{\mu}$$

So we expect the dimensionless pressure Π to be a universal function of the Reynolds number Re:

$$\Pi = f(\mathrm{Re})$$

Experiments bear this out.

The Moody diagram



Symmetry and heat conduction

$$\rho c \frac{\partial u}{\partial t} = k \frac{\partial^2 u}{\partial x^2}$$

$$[\rho c] = \frac{\mathsf{M}}{\Theta \mathsf{L} \mathsf{T}^2}, \quad [k] = \frac{\mathsf{ML}}{\Theta \mathsf{T}^3}, \quad [x] = \mathsf{L}, \quad [t] = \mathsf{T}, \quad [u] = \Theta$$

Just one dimensionless combination:

$$\frac{x^2}{t} \cdot \frac{\rho c}{k}$$

By selecting the length scale L and time scale T so that $X^2/T = k/(\rho c)$ we get the dimensionless form of the heat equation:

$$\frac{\partial u}{\partial t} = \frac{\partial^2 u}{\partial x^2}$$

Symmetry and heat conduction

$$\frac{\partial u}{\partial t} = \frac{\partial^2 u}{\partial x^2}$$

This leaves a degree of freedom still: Rescaling $x = \alpha x^*$, $t = \alpha^2 t^*$ yields the equation invariant.
In addition, we have the obvious scale invariance on u, since the equation is linear.

We combine these to solve the initial value problem with

$$u(x, 0) = \delta(x)$$

If u solves this problem, then so does $\alpha u(\alpha x, \alpha^2 t)$.

Similarity solution

Similarity solution:

$$u(x,t) = \alpha u(\alpha x, \alpha^2 t)$$

for all x, t, α. With $\alpha = 1/\sqrt{t}$ this leads to

$$u(x,t) = \frac{1}{\sqrt{t}} v\Big(\frac{x^2}{t}\Big)$$

and a corresponding ODE for v. The final solution is

$$u(x,t) = \frac{1}{2\sqrt{\pi t}} \exp\Big(-\frac{x^2}{4t}\Big).$$

Sophus Lie (1842–1899)

Lie Symmetries

The geometry of a differential equation for $q = q(t)$:

$$F(t, q, \dot{q}, \ddot{q}) = 0$$

Reduces to a first order system:

$$\mathrm{d}q = \dot{q}\,\mathrm{d}t, \quad \mathrm{d}\dot{q} = \ddot{q}\,\mathrm{d}t \tag{$*$}$$

A vector field in (t, q) space induces a one-parameter group of transformations, which is prolonged to $(t, q, \dot{q}, \ddot{q})$ space by insisting that it respects $(*)$.

If the surface $F = 0$ is invariant: A symmetry group of the ODE.

Symmetries reduce order

Each one-parameter symmetry group allows the reduction of order of the differential equation by 1.

Recall the heat conduction example:
Rescaling symmetries reduce the order.

But not all symmetries are due to rescaling.

Variational problems

$$J(q) = \int_a^b L(t, q, \dot{q})\,\mathrm{d}t$$

How to minimize $J(q)$? Any stationary point of J must satisfy the Euler–Lagrange equation

$$\frac{\partial L}{\partial q} - \frac{\mathrm{d}}{\mathrm{d}t}\frac{\partial L}{\partial \dot{q}} = 0$$

For example, $L(t, q, \dot{q}) = \frac{1}{2}\dot{q}^2 - V(q)$ produces this Euler–Lagrange equation:

$$\ddot{q} = -V'(q)$$

Newton's law for a free particle in a potential field V.

Emmy Noether (1882-1935)

Variational symmetries

These are one-parameter groups which leave the action integral

$$J(q) = \int_a^b L(t, q, \dot{q})\,\mathrm{d}t$$

invariant.

Noether's theorem establishes a one-to-one correspondence betwwen variational symmetries and integrating factors, and hence invariants, of the Euler–Lagrange equations.

Famous conservation laws

Translational symmetry implies the conservation of momentum.

Rotational symmetry implies the conservation of angular momentum.

And time invariance implies the conservation of energy