Exercise Set 2

Problem 1

(a) The perturbation expansion of u is given by

$$u = \sum_{n=0}^{\infty} \varepsilon^n u_n.$$

We plug it in to the governing equation

$$u'' + u = 1 + \varepsilon u^2$$

and get

$$\sum_{n=0}^{\infty} \varepsilon^n u_n'' + \sum_{n=0}^{\infty} \varepsilon^n u_n = 1 + \varepsilon (\sum_{n=0}^{\infty} \varepsilon^n u_n)^2$$

or

$$\sum_{n=0}^{\infty} \varepsilon^n (u_n'' + u_n) = 1 + \varepsilon \sum_{i,j=0}^{\infty} \varepsilon^{i+j} u_i u_j.$$

Equaling the terms of order 0, we get

$$u_0'' + u_0 = 1. \tag{1}$$

Equaling the terms of order n $(n > 0)$, we get

$$u_n'' + u_n = \sum_{\substack{i,j \in \mathbb{N} \\ i+j+1=n}} u_i u_j$$

or

$$u_n'' + u_n = \sum_{i=0}^{n-1} u_i u_{n-1-i}. \tag{2}$$

(b) A general solution of (1) is given by

$$u_0(\theta) = 1 + A \cos\theta + B \sin\theta.$$

The initial conditions $u(0) = e + 1$ and $u'(0) = 0$ imply that $A = e$ and $B = 0$. Hence,

$$u_0(\theta) = 1 + e \cos\theta.$$

From (2), we get the equation satisfied by u_1:

$$u_1'' + u_1 = u_0^2$$

which gives, after replacing u_0,

$$u_1'' + u_1 = (1 + e\cos\theta)^2.$$

We expand the right-hand side and, after using the identity $\cos^2\theta = \frac{1}{2}(\cos 2\theta + 1)$, get

$$u_1'' + u_1 = (1 + \frac{e^2}{2}) + 2e\cos\theta + \frac{e^2}{2}\cos 2\theta. \tag{3}$$

The solution of the homogeneous solution corresponding to (3) is $A\cos\theta + B\sin\theta$. We have to find a particular solution. For the term $\frac{e^2}{2}\cos 2\theta$, a solution of the form $\alpha\frac{e^2}{2}\cos 2\theta$ will do and after some calculation, we get $\alpha = -\frac{1}{3}$. The second term is a bit more tricky since $\cos\theta$ is solution of the homogeneous equation. We use the method of variation of the constant. To find a particular solution of

$$v'' + v = \cos\theta \tag{4}$$

we write v as

$$\begin{aligned} v(\theta) &= \alpha(\theta)\cos\theta + \beta(\theta)\sin\theta & (5) \\ v'(\theta) &= \alpha(\theta)(-\sin\theta) + \beta(\theta)\cos\theta & (6) \end{aligned}$$

where α, β are unknown functions (such functions allways exist because $\cos\theta$ and $\sin\theta$ are two independant solutions of the homogeneous system).

Then, after differentiating (5) and using (6), we get

$$0 = \alpha'(\theta)\cos\theta + \beta'(\theta)\sin\theta. \tag{7}$$

Since v is solution of (4), we also have

$$\cos\theta = \alpha'(\theta)(-\sin\theta) + \beta'(\theta)\cos\theta. \tag{8}$$

Equations (7) and (8) form a two by two system which gives us

$$\begin{aligned} \alpha'(\theta) &= -\sin\theta\cos\theta \\ \beta'(\theta) &= \cos^2\theta \end{aligned}$$

and, after integration,

$$\begin{aligned} \alpha &= \frac{\cos 2\theta}{4} \\ \beta &= \frac{\sin 2\theta}{4} + \frac{\theta}{2}. \end{aligned}$$

Hence, we get

$$\begin{aligned} v &= \alpha(\theta)\cos\theta + \beta(\theta)\sin\theta \\ &= \frac{\cos 2\theta}{4}\cos\theta + (\frac{\sin 2\theta}{4} + \frac{\theta}{2})\sin\theta \\ &= \frac{\theta}{2}\sin\theta + \frac{1}{4}\cos\theta \quad \text{(after some computation).} \end{aligned}$$

Since we are only interested in finding a particular solution, we can drop the $\cos\theta$ term (it is solution of the homogeneous equation) and take $v = \frac{\theta}{2}\sin\theta$.

Finally, the solution of (3) is

$$u_1(\theta) = (1 + \frac{e^2}{2}) + e\theta\sin\theta - \frac{e^2}{6}\cos 2\theta + A\cos\theta + B\sin\theta.$$

The boundary conditions $u_1(0) = 0$ and $u_1'(0) = 0$ give us $A = -(1+\frac{e^2}{3})$, $B = 0$ and

$$u_1(\theta) = (1 + \frac{e^2}{2}) + e\theta\sin\theta - \frac{e^2}{6}\cos 2\theta - (1 + \frac{e^2}{3})\cos\theta.$$

The term $\theta\sin\theta$ is not physical because it grows to infinity. Our approximation is only valid on a small interval when $\theta\sin\theta$ remains of order 1.

(c) We introduce the function v defined as

$$v(\phi) = u(\theta) \tag{9}$$

where

$$\phi = (1 + \varepsilon h)\theta.$$

We differentiate twice (9) and get

$$(1 + \varepsilon h)^2 v''(\phi) = u''(\theta).$$

We plug in this expression in the governing equation:

$$(1 + \varepsilon h)^2 v''(\phi) + v(\phi) = 1 + \varepsilon v^2(\phi). \tag{10}$$

We expand v in a power serie of ε up to the order 1

$$v(\phi) = v_0(\phi) + \varepsilon v_1(\phi) + o(\varepsilon)$$

and from (10) we get

$$(v_0 + \varepsilon v_1)''(1 + \varepsilon h)^2 + v_0 + \varepsilon v_1 = 1 + \varepsilon(v_0 + \varepsilon v_1)^2 + o(\varepsilon).$$

Equaling the terms of same orders we end up with the following equations for v_0 and v_1:

$$v_0'' + v_0 = 1 \tag{11}$$

$$2hv_0'' + v_1'' + v_1 = v_0^2. \tag{12}$$

We have $v_0 = 1 + e\cos\theta$ (v_0 satisfies the same equation with the same boundary conditions as u_0 in the previous question). After some simplification in (12), we get that v_1 satisfies

$$v_1''(\phi) + v_1(\phi) = (1 + \frac{e^2}{2}) + \frac{e^2}{2}\cos 2\phi + 2e(1 + h)\cos\phi. \tag{13}$$

The right-hand side is almost the same as in the previous question and we use the result we found there to get the general solution of (13):

$$v_1(\phi) = (1 + \frac{e^2}{2}) + e(1+h)\phi \sin\phi - \frac{e^2}{6}\cos 2\phi + A\cos\phi + B\sin\phi.$$

We set $h = -1$ so that we get rid of the unphysical term $\phi\sin\phi$. The boundary conditions for v_1 imply that $A = -(1 + \frac{e^2}{3})$ and $B = 0$. We end up with

$$v_1(\phi) = (1 + \frac{e^2}{2}) - \frac{e^2}{6}\cos 2\phi - (1 + \frac{e^2}{3})\cos\phi$$

which is 2π-periodic with respect to ϕ.

(d) The system has period 2π with respect to ϕ. $\phi = 2\pi$ when

$$\begin{aligned} \theta &= \frac{2\pi}{1+\varepsilon h} \\ &= 2\pi(1-\varepsilon h) \quad \text{(at first order in } \varepsilon) \\ &= 2\pi + 2\pi\varepsilon \end{aligned}$$

since $h = -1$.

The perihelion (the point where the planet is the closest to the sun) moves forward with an angle $2\pi\varepsilon$ at each rotation.