Exercise Set 5

Exercise 1 (Logan page 377, problem 2.1d)

Let f denote

$$f(u) = u^2(u^2 - 1).$$

The fixed points of the systems are given by the roots of f:

$$u = \pm 1, 0.$$

In general, for a given root u of f,

- if $f'(u) < 0$ then u is a stable point.
- if $f'(u) > 0$ then u is an unstable point.
- if $f'(u) = 0$ then we have to look at the second derivative. If $f''(u) \neq 0$, u is unstable otherwise we have to look at the next derivative and so forth.

These results are easily proved by looking at the taylor expansion of f around u. To illustrate this, let's look at $u = 0$ for our given function f. We have

$$\frac{du}{dt} = -u^2 + o(u^2)$$

and the system is unstable because if u is a little bit smaller than 0, the expansion above holds and du/dt is strictly negative, u decreases and therefore u goes further away from 0. In this case, we have

$$f'(0) = 0 \quad \text{and} \quad f''(0) = -2.$$

At $u = -1$,

$$f'(-1) = -2 < 0$$

and the system is stable (locally we have $d(u+1)/dt = -2(u+1) + o(u+1)$).

At $u = 1$,

$$f'(1) = 2 > 0$$

and the system is unstable.

Exercise 2 (Logan page 377, problem2.2b)

We apply directly the theory presented in the book of Logan (page 357-377). The equilibrium points are given by the following curves in the (μ, u) plane (see figure below):

$$u = 0$$
$$9 - \mu u = 0$$
$$\mu + 2u - u^2 = 0$$

At $(\mu, u) = (-1, 1)$, $\frac{d\mu}{du}$ changes sign. We have

$$f_\mu(\mu, u) = u(9 - 2u\mu - 2u^2 + u^3).$$

Hence $f_\mu(-1, 1) = 10 \neq 0$ and $(-1, 1)$ is a regular turning point where stability is exchanged.

We differentiate f_μ once more and get

$$f_{\mu\mu} = -2u^2-$$

At (μ_1, u_1), the intersection between the two curves $9 - \mu u = 0$ and $\mu + 2u - u^2 = 0$, $f_u = 0$ but $f_{\mu\mu}$ does not vanish. Thus, we have a double point and stability is exchanged (theorem 2.4, p.370 in Logan).

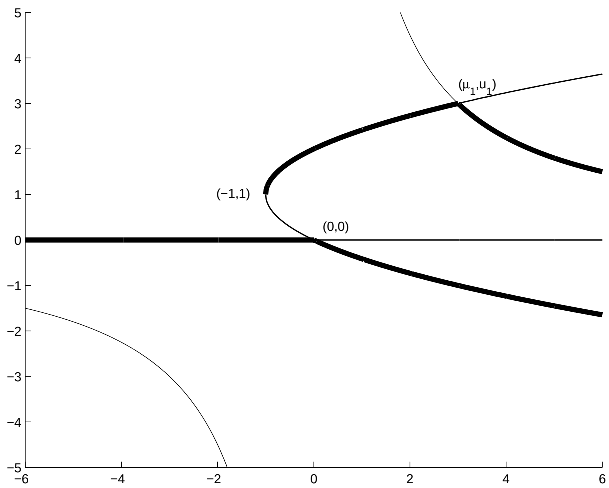
$f_{\mu\mu}(0, 0) = 0$ but $f_{\mu u}(0, 0) = 9 \neq 0$. $(0, 0)$ is a double point and stability is also exchanged (theorem 2.5, p.371 in Logan).

It then suffices to compute the sign of f_u at one point of each curve to determine the stability along all the curves. We have

$$f_u = (9 - \mu u)(\mu + 2u - u^2) - \mu u(\mu + 2u - u^2) + u(9 - \mu u)(-2u + 2)$$

We choose for example (μ, u) equal to $(0, 2)$, $(0, -\infty)$, $(\mu, \frac{9}{\mu})$ $\mu \to \infty$. We get

$$f_u(0, 2) = -180$$
$$\lim_{\mu \to -\infty} f_u(0, u) = -\infty$$
$$\lim_{u \to \infty} f_u(\mu, 9\frac{9}{\mu}) = -\infty$$



The thick lines indicate stable equilibrium points