Exercise Set 7

Problem 1

(a) From Newton's second law of motion, we have that, given a body of mass M,

$$M\underline{a} = \sum \underline{F}$$

where $\underline{a}$ is the acceleration of the body and $\sum \underline{F}$ is the sum of all the forces acting on that body.

In the case of the sprinter, the forces we will consider are:

- F_s: The actual force produced by the sprinter.
- F_i: A single force which accounts for all the friction forces acting within the body of the sprinter.
- F_D: The drag produced by the air (it has to be added with a negative since the effect of the drag is to slow down the sprinter).

Hence,

$$M\frac{du^*}{dt^*} = F_s + F_i - F_D.$$

As indicated in the text, we write

$$F_s = Mp^*(t^*) \quad \text{and} \quad F_i = -MR(u^*) = -\frac{Mu^*}{\tau}.$$

It remains to evaluate F_D. In this purpose, we proceed to a dimensional analysis. The drag F_D depends on the speed of the sprinter. It depends on his general shape but we assume that all the sprinters have roughly the same shape and the only paramater which matters is the surface of their "cross-section":A. Concerning the air, ν and ρ have to be considered (the heavier is the air, the more energy is needed to displace it). We get the following dimension table:

	A	ρ	u^*	ν	F_D
kg	0	1	0	0	1
m	2	-3	1	2	1
s	0	0	-1	-1	-2

The rank of the system is 3. We get 5-3=2 independant variables:

$$\Pi_1 = \frac{\nu}{u^* A^{1/2}},\ \Pi_2 = \frac{F_D}{\rho A u^{*2}}.$$

Hence, by the Buckingham's pi theorem,

$$\frac{F_D}{\rho A u^{*2}} = \Phi(\frac{\nu}{u^* A^{1/2}}).$$

After substituting $A^{1/2}$ by a characteristic length L, we can rewrite this formula as given in the text:

$$F_D = \frac{1}{2}\rho C_D(R_e)Au^{*2}$$

where $C_D(x) = 2\Phi(1/x)$.

(b) We want to determine the scales $\overline{u}, \overline{t}, \overline{p}$ defined as

$$u^* = \overline{u}u, \ t^* = \overline{t}t, \ p^* = \overline{p}p.$$

After plugging these expressions into the equation of motion, we get

$$M\frac{\overline{u}}{\overline{t}}\frac{du}{dt} + \frac{M\overline{u}}{\tau}u + \frac{1}{2}\rho C_D A\overline{u}^2u^2 = M\overline{p}p. \tag{1}$$

To get the desired form, the terms in front of $\frac{du}{dt}$, u and p have to be equal:

$$M\frac{\overline{u}}{\overline{t}} = \frac{M\overline{u}}{\tau} = M\overline{p}.$$

It follows

$$\overline{t} = \tau \ \text{ and } \ \overline{u} = \tau\overline{p}.$$

A natural choice for $\overline{p}$ is $P = \max(p)$ and we end up with $\overline{u} = \tau P$. Equation (1) becomes

$$\dot{u}(t) + u(t) + \varepsilon u(t)^2 = p(t) \tag{2}$$

where $\varepsilon = \frac{1}{2}\rho C_D \frac{A}{M}\tau^2 P$.

If we take $M = 80kg$, we get $\varepsilon = 2\%$ and the resistance of the air can be neglected in a first approximation.

(c) We expand $u(t)$ up to the first order

$$u = u_0 + \varepsilon u_1 + o(\varepsilon).$$

Plugging this expression into equation (2), we get

$$\dot{u}_0 + \varepsilon\dot{u}_1 + u_0 + \varepsilon u_1 + \varepsilon(u_0 + \varepsilon u_1 + o(\varepsilon))^2 = p(t) + o(\varepsilon).$$

We expand the quadratic term and ignore the terms of order larger than ε.

$$\dot{u}_0 + u_0 + \varepsilon(\dot{u}_1 + u_1 + u_0^2) = p(t) + o(\varepsilon)$$

The initial condition is

$$u(0) = u_0(0) + \varepsilon u_1(0) + o(\varepsilon) = 0.$$

At the order 0, we get

$$\dot{u}_0 + u_0 = p(t), \ u_0(0) = 0. \tag{3}$$

At the order 1, we get

$$\dot{u}_1 + u_1 + u_0^2 = 0, \;\; u_1(0) = 0. \tag{4}$$

We set $p(t) = 1$ in (3) ($p^* = P$ implies $p = 1$). We have to solve

$$\dot{u}_0 + u_0 = 1. \tag{5}$$

A particular solution of (5) is $u_0 = 1$ while the solution of the homogeneous equation is $u_0 = e^{-t}$. Hence, the general solution of (5) is $u_0(t) = Be^{-t} + 1$ (B =constant). $u_0(0) = 0$ implies that $B = -1$ and we finally have

$$u_0(t) = 1 - e^{-t}.$$

Equation (4) becomes

$$\dot{u}_1 + u_1 + (1 - e^{-t})^2 = 0$$

or

$$\dot{u}_1 + u_1 = -1 + 2e^{-t} - e^{-2t}. \tag{6}$$

A particular solution of (6) is $-1 + 2te^{-t} + e^{-2t}$, the general solution is $Be^{-t} - 1 + 2te^{-t} + e^{-2t}$ (B =constant). $u_1(0) = 0$ implies that $B = 0$ and we have

$$u_1(t) = -1 + 2te^{-t} + e^{-2t}.$$

Finally we get

$$u(t) = 1 - e^{-t} + \varepsilon(-1 + 2te^{-t} + e^{-2t}) + o(\varepsilon)$$

and

$$\lim_{t\to\infty} u = 1 - \varepsilon + o(\varepsilon).$$

(d) If we take into account the effect of the wind, the equation of motion (2) becomes

$$\dot{u} + u + \varepsilon(u - \delta)^2 = p(t).$$

Then u_0 and u_1 satisfy

$$\begin{aligned} \dot{u}_0 + u_0 &= 1 \\ \dot{u}_1 + u_1 + (u_0 - \delta)^2 &= 0. \end{aligned}$$

A calculation very similar to the case without wind gives us

$$u_1 = -(1 - \delta)^2 + \delta(\delta - 2)e^{-t} + 2(1 - \delta)te^{-t} + e^{-2t}$$

and

$$\lim_{t\to\infty} u = 1 - \varepsilon(1 - \delta)^2.$$

(e) In the case of Florence Griffith-Joyner, we take $M = 60kg$. We compute δ and ε:

$$\delta = 0.4 \text{ and } \varepsilon = 0.038.$$

The maximal speed with the wind (u_w^m) is

$$u_w^m = 1 - \varepsilon(1-\delta)^2 = u^m + \varepsilon\delta(2-\delta)$$

where u^m is the maximal speed without wind. Hence,

$$\frac{u_m^w - u^m}{u^m} = \frac{\varepsilon\delta(2-\delta)}{1-\varepsilon} = 2.5\%.$$

The performance of the sprinter is improved by 2.5%. If we apply the same ratio to the average performance of the sprinter during the same period ($10.7s$) we get

$$t = 10.43s$$

which gives an approximation of the effect of the wind on the sprinter performance. It is fairly close to the actual result of the sprinter.

Problem 2

If y satisfies the differential equation with the given boundary conditions, so does $y(1-x)$. The problem is symmetric with respect to $x = 1/2$ and we will focus our attention on the boundary layer on the left, around 0.

The inner expansion y_m of y satisfies

$$y_m'' + \lambda y_m = 0$$

which gives

$$y_m = A\cos\sqrt{\lambda}x + B\sin\sqrt{\lambda}x \tag{7}$$

where A and B are constant.

Around 0, we rescale the problem with $x_l = \frac{x}{\varepsilon^\alpha}$ and $y_l(x_l) = y(x)$. We get

$$\varepsilon^{1-4\alpha} y_l'''' - \varepsilon^{-2\alpha} y_l'' = \lambda y_l.$$

We take $\alpha = 1/2$ so that the fourth derivative is taken into account. At the lowest order, we have

$$y_l'''' - y_l'' = 0. \tag{8}$$

The general solution of this equation is

$$y_l = Ax_l + B + Ce^{x_l} + De^{-x_l}.$$

The terms $x_l = x/\sqrt{\varepsilon}$ and $e^{x_l} = e^{\frac{x}{\sqrt{\varepsilon}}}$ are not possible, since we (implicitly) assumed while doing our expansion that $y_l = O(1)$. We are left with

$$y_l = B + De^{-x_l}.$$

Taking into account the boundary conditions $y_l(0) = y_l'(0) = 0$, we get that $B = D = 0$ and

$$y_l = 0. \tag{9}$$

A similar calculation on the right, around $x = 1$, would give us

$$y_r = 0.$$

We can set the constants A and B in (7) by matching y_m with y_l and y_r. We have

$$\lim_{x \to 0} y_m(x) = \lim_{x_l \to \infty} y_l(x_l)$$

which implies, since $y_l = 0$,

$$A = 0$$

and

$$\lim_{x \to 1} y_m(x) = \lim_{x_r \to -\infty} y_r(x_r)$$

which implies

$$B \sin \sqrt{\lambda} = 0.$$

We take $B \neq 0$ (we exclude the zero solution which is not a eigenfunction) and we arbitrarily set $B = 1$ since any multiple of a solution remains solution for the same eigenfrequency. We have:

$$\lambda = \pi^2 n^2, \quad n \in \mathbb{N}\backslash\{0\}.$$

At the left-hand side, $y_l = 0$ does not give us a satisfactory picture of the solution. y_l has to grow at some point in order to match with $y_m = \sin \sqrt{\lambda} x$. The fact is that y_l is not only $O(1)$ but $O(\sqrt{\varepsilon})$ as we will now see. Let's introduce y_{l1} ($y_{l1} = O(1)$) defined as

$$y_{l1} = \sqrt{(\varepsilon)} y_{l1} + o(\sqrt{\varepsilon}). \tag{10}$$

y_{l1} satisfies (8) and we have, as earlier,

$$y_{l1} = A x_l + B + C e^{x_l} + D e^{-x_l}.$$

When ε tends to infinity, the term $x_l = x/\sqrt{\varepsilon}$ is compensated by the factor $\sqrt{\varepsilon}$ in front of y_{l1} in equation (10). Therefore A does not have to vanish. The term e^{x_l} however still goes to infinity and we must impose $C = 0$. The boundary conditions $y_{l1}(0) = y_{l1}'(0) = 0$ imply that $A = -B = D$ and y_{l1} takes the form

$$y_{l1} = A(x_l - 1 + e^{-x_l}).$$

In order to match y_l with y_m, we introduce the intermediate scaling $\hat{x} = \varepsilon^{\beta} x_l = \varepsilon^{\beta - 1/2} x$ where $\beta \in (0, \frac{1}{2})$. We rewrite y_l and y_m in terms of $\hat{x}$:

$$\begin{aligned} y_l &= A(\varepsilon^{1/2-\beta}\hat{x} - \varepsilon^{1/2} + \varepsilon^{1/2} e^{-\varepsilon^{-\beta}\hat{x}}) \\ y_m &= \sin\left(\sqrt{\lambda}\varepsilon^{1/2-\beta}\hat{x}\right) = \sqrt{\lambda}\varepsilon^{1/2-\beta}\hat{x} + o(\varepsilon^{1/2-\beta}). \end{aligned}$$

We equal the lowest order terms and get

$$A = \sqrt{\lambda}$$

and the outer expansion on the left looks like

$$y_l = \sqrt{\lambda}(x - \sqrt{\varepsilon} + \sqrt{\varepsilon} e^{-\frac{x}{\sqrt{\varepsilon}}}).$$