Exercise Set 9

Exercise 1

(a) We write up the total energy E for the system. It consists of the sum of the kinetical energy

$$K = \frac{1}{2}mv^2$$

and the potential energy

$$W = mgy.$$

Hence,

$$\begin{aligned} E &= K + W \\ &= \frac{1}{2}mv^2 + mgy. \end{aligned}$$

Note that we have dropped the asterisks. We will reintroduce them later when we rescale the equation. The velocity v is given by $v = [\dot{x}(t), \dot{y}(t)]$. Since $y(t) = \frac{x(t)^2}{2b}$, we get

$$v = \left[\dot{x}(t), \frac{\dot{x}(t)x(t)}{b}\right]$$

and

$$E = \frac{1}{2}m\left(\dot{x}^2 + \frac{\dot{x}^2x^2}{b^2}\right) + mg\frac{x^2}{2b}. \tag{1}$$

Since there is no friction, the total energy is conserved i.e. $\frac{dE}{dt} = 0$. After differentiating (1) and dividing by $m\dot{x}$, we get

$$\ddot{x}\left(\frac{x^2}{b^2} + 1\right) + \dot{x}^2\frac{x}{b^2} + \frac{gx}{b} = 0 \tag{2}$$

which is the equation of motion. A natural scaling for x^* is $x^* = ax$. Let $t^* = Tt$. we have, from (2) after having reintroduced the asterisks,

$$\frac{a}{T^2}\ddot{x}\left(\frac{a^2}{b^2}x^2 + 1\right) + \frac{a^3}{T^2b^2}x + \frac{a}{b}gx = 0$$

we balance the first term which involves the second derivative of x with the last one which involves the gravity by setting

$$\frac{a}{T^2} = \frac{a}{b}g$$

or

$$T = \sqrt{\frac{b}{g}}.$$

We end up with the following equation:

$$\left(\varepsilon x^2 + 1\right)\ddot{x} + \varepsilon x \dot{x}^2 + x = 0 \tag{3}$$

and the initial conditions are $x(0) = 1$, $\dot{x}(0) = 0$.

(b) At the zero order, equation (3) gives us

$$\dot{x}_0 + x_0 = 0. \tag{4}$$

The general solution of (4) is $x_0 = A\cos t + B\sin t$. The constants A and B are determined by the initial conditions. We have:

$$x_0(t) = \cos t.$$

At the first order, we have

$$\ddot{x}_1 + x_1 = -x_0\dot{x}_0^2 - x_0^2\ddot{x}_0$$

or, after replacing x_0 by $\cos t$,

$$\ddot{x}_1 + x_1 = -\cos t \tag{5}$$

The solution of (5) is given by $x_1 = -\frac{1}{2}t\sin t + A\cos t + B\sin t$. The constants A and B are equal to 0 because of the initial conditions. Hence,

$$x_1 = -\frac{1}{2}t\sin t.$$

x_1 is unbounded and therefore the assumption $x_1 = O(1)$ fails after some time.

As suggested in the text, we introduce the auxiliary time variable

$$\tilde{t} = (1 + \varepsilon c)t$$

We set $\tilde{x}(\tilde{t}) = x(t)$. It implies that

$$\begin{aligned} \frac{dx}{dt} &= (1 + \varepsilon c)\frac{d\tilde{x}}{d\tilde{t}} \quad \text{and} \\ \frac{d^2x}{dt^2} &= (1 + \varepsilon c)^2\frac{d^2\tilde{x}}{d\tilde{t}}. \end{aligned}$$

Plugging that into (3), we get an equation for the $\tilde{x}$ and $\tilde{t}$. The equation for $\tilde{x}_0$ is the same as for x_0 and therefore we get

$$\tilde{x}_0 = \cos\tilde{t}.$$

At the first order, we get

$$\ddot{\tilde{x}}_1 + \tilde{x}_1 = -2c\dot{\tilde{x}}_0 - \tilde{x}_0^2\ddot{\tilde{x}}_0 - \tilde{x}_0\dot{\tilde{x}}_0^2$$

which leads to

$$\ddot{\tilde{x}}_1 + \tilde{x}_1 = (2c-1)\cos\tilde{t}.$$

If we take $c = \frac{1}{2}$, the secular term disappear and $\tilde{x}_1 = 0$. The point particle is oscillating between $x = -1$ and $x = 1$ with a period $1 + \varepsilon c$ at the first order in ε.

Exercise 2

(a) The variation of the population of healthy people which are not immuned is given by

$$\frac{dx^*}{dt^*} = -\beta_1 x^* y^* + \beta_2 y^* - \beta_3 \tag{6}$$

where β_1, β_2 and β_3 are constants. In (6) the first term, $-\beta_1 x^* y^*$, corresponds to the people that get sick. Since the disease is an infectious disease, the *chance* of getting ill increases with the number of sick people, y, and that's why we get this nonlinear term. The second term corresponds to the people that recover whithout becoming immune while the last term corresponds to people that are vaccinated and therefore become immuned. The variation of the population of sick people is given by

$$\frac{dy^*}{dt^*} = \beta_1 x^* y^* - \beta_4 y^*. \tag{7}$$

The first term in (7) is the counterpart of the first term in (6). The second term corresponds to people that recover. They can become immune or not and it implies that $\beta_4 \geq \beta_2$. The variation of the population of immuned people is given by

$$\frac{dz^*}{dt^*} = (\beta_4 - \beta_2) y^* + \beta_3 \tag{8}$$

and is obtained from the fact that the total population is constant:

$$\frac{dz^*}{dt} = -\frac{dx^*}{dt^*} - \frac{dy^*}{dt^*}.$$

A natural scaling for x^*, y^* and z^* is the total population P. After rescaling time as $t^* = \frac{1}{\beta_4} t$, we get

$$\dot{x} = -\alpha xy + \varepsilon y - \kappa \tag{9}$$

$$\dot{y} = \alpha xy - y. \tag{10}$$

for some dimensionless constants α, ε and κ.

(b) The constraints are that x, y and z are positive and their sum is constant: $x^* + y^* + z^* = P$ which implies that $x + y + z = 3$. It follows that we must have

$$\begin{cases} x \geq 0 \\ y \geq 0 \\ x + y \leq 3 \end{cases}$$

If there is no vaccination, $\kappa = 0$ and we get:

$$\dot{y} = y(\alpha x - 1) \tag{11}$$
$$\dot{x} = (\varepsilon - \alpha x)y. \tag{12}$$

Dividing (11) by (12), we get

$$\frac{\dot{y}}{\dot{x}} = \frac{dy}{dx} = \frac{\alpha x - 1}{\varepsilon - \alpha x}.$$

We can integrate this expression:

$$\frac{dy}{dx} = -1 + \frac{\varepsilon - 1}{\varepsilon - \alpha x}$$

implies

$$y = -x + \frac{1 - \varepsilon}{\alpha} \ln(|\alpha x - \varepsilon|) + C. \tag{13}$$

Equation (13) is an expression for the orbit when nobody is vaccinated.

(c) From (10), we get that $\dot{x}$ vanishes on the curve

$$y = \frac{\kappa}{\alpha\left(\frac{\varepsilon}{\alpha} - x\right)}$$

while $\dot{y}$ vanishes when

$$y = 0 \text{ and } x = \frac{1}{\alpha}.$$

In figure 1, some orbits are plotted for $\kappa = 0.1$, $\varepsilon = 0.5$ and $\kappa = 1.5$.

(d) The critical point (x_c, y_c) satisfies

$$0 = -\alpha x_c y_c + \varepsilon y_c - \kappa,$$
$$0 = \alpha x_c y_c - y_c.$$

Hence,

$$x_c = \frac{1}{\alpha} \quad \text{and } y_c = \frac{\kappa}{\varepsilon - 1}.$$

At the critical point the linear approximation of (9) and (10) is given by the matrix A:

$$A = \begin{pmatrix} -\frac{\alpha\kappa}{\varepsilon - 1} & \varepsilon - 1, \\ \frac{\alpha\kappa}{\varepsilon - 1} & 0. \end{pmatrix}$$

Let λ_1 and λ_2 be the two eigenvalues of A. We have

$$\lambda_1 + \lambda_2 = \mathrm{tr}A = -\frac{\alpha\kappa}{\varepsilon - 1} > 0$$

and

$$\lambda_1 \lambda_2 = \mathrm{det}A = -\alpha\kappa < 0.$$

Hence, λ_1 and λ_2 have opposite sign and the system is unstable.

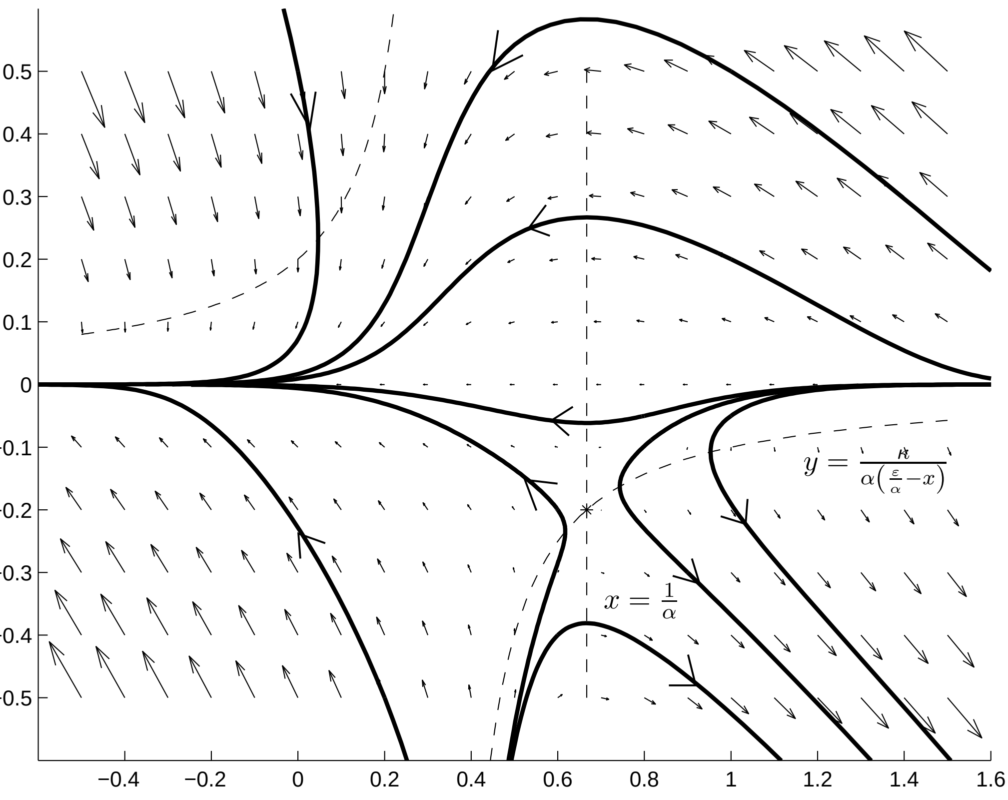

Figure 1: phase plane