Suggested solution for exercise Set B

The dimension matrix for the involved quantities, remembering the unusual units for F (it is force *per unit length* along the river):

	F	b	u	ρ	ν
kg	1	0	0	1	0
m	0	1	1	−3	2
s	−2	0	−1	0	−1

The matrix has rank 3, so there are 5 – 3 = 2 independent dimensionless combinations – for example

$$\mathrm{Re} = \frac{bu}{\nu} \quad \text{and} \quad \frac{F}{b\rho u^2}.$$

Any relation between these quantities should therefore be rewritable as an equation involving only these two combinations. Assuming that F is a function of the other quantities, then, we may write

$$\frac{F}{b\rho u^2} = f(\mathrm{Re}), \quad \text{or in other words } F = f(\mathrm{Re})\, b\rho u^2.$$

For F to be linear in b, it must be the case that $f(\mathrm{Re})$ is constant. We call this constant the *friction factor* and write

$$F = Cb\rho u^2.$$

We could have gotten here quicker by starting with the variables F/b, u, ρ and ν.

From the force balance equation $F = bgh\rho \sin\alpha$ and the above equation we get $Cu^2 = gh\sin\alpha$ (after division by $b\rho$), which we write

$$u = C' h^{1/2}, \qquad \text{where } C' = \sqrt{\frac{g\sin\alpha}{C}}.$$

The total (volume) flux is bhu, while $\int_{x_1}^{x_2} bh\,dx$ gives the volume of water between positions x_1 and x_2 along the river. So we find the volume conservation law:

$$\frac{d}{dt}\int_{x_1}^{x_2} bh\,dx + \Big[bhu\Big]_{x_1}^{x_2} = 0.$$

We divide by the (assumed!) constant b, and substitute the above value of u to get

$$\frac{d}{dt^*}\int_{x_1^*}^{x_2^*} h^*\,dx^* + \Big[C' h^{*3/2}\Big]_{x_1^*}^{x_2^*} = 0,$$

where I have now added asterisks in order to prepare the equation for scaling. With

$$h^* = Hh, \quad x^* = Xx, \quad t^* = Tt$$

we get

$$\frac{HX}{T}\frac{d}{dt}\int_{x_1}^{x_2} h\,dx + C'H^{3/2}\Big[h^{3/2}\Big]_{x_1}^{x_2} = 0.$$

Our scaling should be chosen so that

$$\frac{X}{H^{1/2}T} = C',$$

and we arrive at the final, scaled form

$$\frac{d}{dt}\int_{x_1}^{x_2} h\,dx + \Big[h^{3/2}\Big]_{x_1}^{x_2} = 0. \tag{1}$$

If $h(x,t)$ is sufficiently smooth, we can take the derivative inside the first integral and rewrite the second term into the integral of $\left(h^{3/2}\right)_x$, and so we arrive at the differential form

$$h_t + \left(h^{3/2}\right)_x = 0. \tag{2}$$

So why do we get this seemingly ridiculous result, that the flow rate does not depend on viscosity? This makes no sense if we were to fill the river with really thick syrup. The answer is that we have neglected the water depth h in the dimensional analysis. If we did include it, we would get another dimensionless quantity h/b in the equations, and so we should have

$$u = f\Big(\mathrm{Re}, \frac{h}{b}\Big) b\rho u^2.$$

Now, $f(\mathrm{Re}, h/b)$ need not be constant; it only needs to be independent of b, which we can achieve by assuming it depends only on the product of its arguments: $\mathrm{Re}h/b = hu/\nu$ (which is the Reynolds number based on the *depth* of the river, rather than on its width).

Now we have cleared *that* up, another question raises its ugly head: Why was it okay to ignore h when viscosity is low? The answer lies deep in the fluid mechanics, but the short answer is that one typically gets a flow where the average velocity is the same throughout the most of the river, except near the bottom where there will be a turbulent boundary layer. It is the thickness of this boundary layer that is really important, but it will be independent of the water depth (so long as the water is not *too* shallow).

We are looking at the boundary- and initial value problem consisting of (1) (which can be written (2) where h is smooth) and the conditions

$$h(x,0) = 1, \quad h^{3/2}(0,t) = q(t) \qquad \text{where } x > 0 \text{ and } t > 0.$$

The characteristic speed is $d\big(h^{3/2}\big)/dh = \frac{3}{2}h^{1/2}$.

Characteristics starting on the positive x-axis (at $t = 0$) all have $h = 1$, so their speed is $\frac{3}{2}$.

Characteristics starting on the positive t-axis (at $x = 0$) have $h = q^{2/3}$, so their speed is $\frac{3}{2}h^{1/2} = \frac{3}{2}q^{1/3}$. More precisely, we write

$$c(\tau) = \tfrac{3}{2}q^{1/3}(\tau)$$

for the speed of the characteristic starting at $x = 0$, $t = \tau$. The equation of this characteristic is

$$x = c(\tau)\cdot(t-\tau) = \tfrac{3}{2}q^{1/3}(\tau)\cdot(t-\tau).$$

Clearly, if c increases with τ anywhere, characteristics will collide, and shocks must form. This corresponds to the case where the flow rate q increases with time.

Furthermore, if $c(\tau) > \frac{3}{2}$ for any τ, then the corresponding must collide with one of the characteristics from the x-axis – if it is not destroyed by hitting a shock before it gets that far.

If q starts out smaller than 1 and never increases – and only in that case – the characteristics will move apart, and no shock will form.

To find out when characteristics collide, note that to find which characteristic from the t-axis passes through (x, t) we have to solve the equation $x = c(\tau)\cdot(t-\tau)$ with respect to τ. Once the equation is solved, we find h from $h^{3/2} = q(\tau)$.

To analyse the equation, it is easiest to write it in the form

$$t = \tau - \frac{x}{c(\tau)}.$$

When x is small, we can hope that this defines t as an increasing function of τ, so we can invert this and find τ as a function of t (and x). To check if the function *is* increasing, differentiate it:

$$\frac{dt}{d\tau} = 1 - \frac{xc'(\tau)}{c(\tau)^2}.$$

This is positive so long as $xc'/c^2 < 1$. But if $c' > 0$ somewhere, this expression will become equal to 1 for some x. The smallest x for which this happens will be the inverse of the maximum value of $c'/c^2 = \frac{2}{9}q^{-4/3}q'$.