

# TMA4195 Mathematical modelling 2004

## Exercise set 2

Advice and suggestions: 2004–08–31

This problem is a famous exam problem from 1989, when prof. Ola Bratteli (now at the University of Oslo) taught the class. The problem leads to relatively complicated expressions. It might be an advantage to use Maple to keep track. This text contains a few extra hints.

The method used in points c–d is called *Poincaré's method*. It is essentially the same method I used for the movement of a pendulum in the lecture on Thursday.

**Exercise 1:** (Exam 1989, somewhat rephrased).

The relativistic equation for a planet moving around the Sun is, in dimensionless form,

$$\frac{d^2 u}{d\theta^2} + u = 1 + \varepsilon u^2,$$

where  $u = 1/r$ , and  $r$  and  $\theta$  are polar coordinates. The term  $\varepsilon u^2$  is a relativistic correction,  $0 < \varepsilon \ll 1$ .

(a) Show that the  $n$ -th order term in the perturbation expansion

$$u = u_0(\theta) + \varepsilon u_1(\theta) + \varepsilon^2 u_2(\theta) + \dots$$

satisfies a differential equation on the form

$$\frac{d^2 u_n}{d\theta^2} + u_n = f_n(u_0, u_1, \dots, u_{n-1}),$$

where  $f_0$  is a constant and  $f_n$  is a function of  $n$  variables. Find  $f_n$  for all  $n$ .

(Hint: You are not supposed to solve the equations, only to find them!)

(b) Assume the initial conditions

$$u(0) = e + 1, \quad \frac{du}{d\theta}(0) = 0.$$

Here  $e$  is the eccentricity of the unperturbed orbit. With these starting conditions  $\theta = 0$  corresponds to the *perihelion* of the orbit, i.e., the point of the orbit nearest the Sun ( $u = 1/r$  is greatest). Find  $u_0$  and  $u_1$  using regular perturbation using the results from point a).

(Hint: Remember that terms on the righthand side which also are solutions of the homogeneous equation lead to special solutions, such as secular terms of the form  $\theta \cos \theta$  or  $\theta \sin \theta$ . These are unphysical because they have unlimited growth.)

(c) To get rid of secular terms we will also develop the angle in a perturbation series. Introduce a new angular variable

$$\varphi = (1 + \varepsilon h)\theta$$

and let  $v(\varphi) = u(\theta)$ . Here  $h$  is a (for now, undetermined) constant. Transform the equation and the starting conditions so that they involve only  $v$  and  $\varphi$ . Show that there exists a value for  $h$  such that  $v_0$  and  $v_1$  in the new perturbation series

$$v = v_0(\varphi) + \varepsilon v_1(\varphi) + \varepsilon^2 v_2(\varphi) + \dots$$

are periodic with period  $2\pi$  with respect to  $\varphi$ , and find  $v_0$  and  $v_1$ .

(d) Use the result from c) to show that the perihelion of the planet moves forward an angle  $2\pi\varepsilon$  per orbit (to first order in  $\varepsilon$ ).