TMA4195 Mathematical modelling 2004

Exercise set B

Advice and suggestions: 2004–10–26 and 2004–11–02

*This is the second of two exercises that will count 10% towards your final grade. Submit your answer no later than **Monday, 8 November** in Norwegian or English. For more information, see the course's web page.*

Consider a wide, shallow river. Water flowing down the river is subject to a friction force from the bottom. Write F for the total friction force acting on the water, measured *per unit length* of the river. (Thus, the units of F are $\mathrm{N/m} = \mathrm{kg/s^2}$.) We expect F to be a function of the river width b, the average flow velocity u of the water, and the properties of the water itself: Its density rho, and its kinematic viscosity ν (with $[\nu] = \mathrm{m^2/s}$).

It seems reasonable to assume that the dependence of F on b is linear – in other words, that F/b is a function of u, ρ and ν. What does dimensional analysis tell us about this function?

Let the river slope downstream at a constant angle $\alpha \ll 1$. To maintain steady flow, we need a force balance $F = bh\rho g \sin\alpha$, where bh is the cross-sectional area of the river (h is the average river depth).

It is common to measure the flow rate of rivers as volume per second. Conclude that the flow rate of the river is proportional to $h^{3/2}$. (For this conclusion it is important that b does not change (much) when the water lever changes. Imagine a roughly rectangular river channel, with a wide bottom and short, steep sides.) What is the constant of proportionality? (It will contain an unknown, dimensionless constant, of course.) Perhaps it will come as a surprise that the viscosity does not appear in the answer.

Clearly, this is nonsense if the river is filled with syrup instead of water. Can you see where we went wrong in this case?

Write an equation stating that water in the river is conserved (assuming that neither evaporation nor rain, nor the addition of water from tributaries, have any effect). Write this equation on integral form, then change scales and show that the non-dimensionalized form, when converted to a partial differential equation, becomes

$$h_t + \left(h^{3/2}\right)_x = 0.$$

Here, x is a coordinate measuring distance along the river, increasing in the downriver direction. Note that the appropriate velocity scaling will depend on a typical river depth (which corresponds to $h = 1$ after scaling).

Imagine that water is introduced at the head of the river ($x = 0$) at some variable rate $q(t)$ and flows downriver from there. Initially, the entire river contains water at standard depth (say, $h = 1$ for $x > 0$ and $t = 0$). Briefly discuss the formation of shocks (flood waves?) in different situations: Increasing $q(t)$, decreasing $q(t)$.

In the above discussion, I have not put asterisks (as in h^*) even on quantities that we wish to scale later. Those asterisks are a nuisance, so after people have got used to scaling, they commonly ignore them. The danger is that you may end up mixing scaled and unscaled quantities and thus making a big mess of things. So this must be practiced with great caution. It usually helps to first derive the equations (using no asterisks), then adding asterisks where needed, and then doing the scaling, never again referring to the unscaled quantities – except when conclusions need to be drawn in physical terms. In other words, be careful to separate the part of the text where you work with scaled quantitites from the part where you work with unscaled (physical) ones.

Many texts – particularly those of British origin – use the exact opposite conventions to ours: Unscaled quantitities have no asterisks, scaled quantitities do. But then, once scaling is completed, one drops the asterisks once more.