Similarity solutions for the heat equation

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Consider the heat equation in one space dimension:

$$\partial_t u = \partial_x^2 u. \tag{1}$$

Note that the function $(x,t) \mapsto Bu(Ax, A^2 t)$ solves the equation if u does. If u is a non-zero solution satisfying

$$u(x,t) = A^{-\mu} u(Ax, A^2 t) \quad \text{for all } A > 0$$

then u is called a *similarity solution* of the heat equation.[1] By selecting $A = x^{-1}$ one arrives at a representation of a similarity solution in terms of a function of a single variable:[2]

$$u(x,t) = x^{\mu} v\Big(\frac{x^2}{4t}\Big). \tag{2}$$

Now, it is a simple exercise to show that a function u defined in this way solves (1) if and only if $v = v(\xi)$ solves

$$\xi^2 v'' + \Big(\xi^2 + \frac{2\mu+1}{2}\xi\Big) v' + \frac{\mu(\mu-1)}{4} v = 0.$$

Two interesting special cases occur for $\mu \in \{0,1\}$. In these cases, the final term in the above equation drops out, and we are left with a first order separable equation for $w = v'$, with solution given by

$$\int \frac{\mathrm{d}w}{w} = -\int \frac{\xi^2 + (\mu+\frac12)\xi}{\xi^2}\,\mathrm{d}\xi = -(\mu+\frac12)\ln\xi - \xi + \text{constant}$$

so that

$$v'(\xi) = w(\xi) = \text{constant}\cdot \xi^{-\mu-1/2} e^{-\xi}.$$

This latter expression is easily integrated using the substitution $\xi = \eta^2$:

$$\int \xi^{-\mu-1/2} e^{-\xi}\,\mathrm{d}\xi = 2\int \eta^{-2\mu} e^{-\eta^2}\,\mathrm{d}\eta. \tag{3}$$

[1] You may verify that, if $u(x,t) = B(A)u(Ax, A^2 t)$ for all $x > 0$, $t > 0$, and $A > 0$ with B a continuous function of A, we must have $B = A^{-\mu}$ for some μ: First show that $B(A_1 A_2) = B(A_1)B(A_2)$.

[2] I planted the extra factor 4 in the numerator because it does simplify things later. We might also select $A = t^{-1/2}$, leading to the representation $u(x,t) = t^{\mu/2} w(x^2/4t)$, which is of course essentially equaivalent. But our current choice turns out to make the calculations a bit easier.

Heating by constant surface temperature: $\mu = 0$. When $\mu = 0$ the above integral is easily evaluated, leading to

$$v(\xi) = C_1 \operatorname{erf}\eta + C_2 = C_1 \operatorname{erf}\sqrt{\xi} + C_2.$$

With $u(x,t) = v(x^2/4t)$ we can impose the boundary conditions $v(0) = 1$, $v(\infty) = 0$, which imply $C_2 = 1$ and $C_1 + C_2 = 0$. Thus

$$u(x,t) = \operatorname{erfc}\frac{x}{2\sqrt{t}}$$

solves (1) with the initial and boundary conditions

$$u(x,0) = 0, \quad u(0,t) = 1.$$

Heating by constant surface heat flow: $\mu = 1$. With $\mu = 1$ we can evaluate (3) using partial integration:

$$\int \eta^{-2} e^{-\eta^2}\,\mathrm{d}\eta = -\eta^{-1} e^{-\eta^2} - 2\int e^{-\eta^2}\,\mathrm{d}\eta = -\eta^{-1} e^{-\eta^2} - 2\operatorname{erf}\eta.$$

Thus we get

$$v(\xi) = C_1(\eta^{-1} e^{-\eta^2} + 2\operatorname{erf}\eta) + C_2 = C_1(\xi^{-1/2} e^{-\xi} + 2\operatorname{erf}\sqrt{\xi}) + C_2$$

leading to

$$u(x,t) = xv\Big(\frac{x^2}{4t}\Big) = 2C_1\Big(\sqrt{t}e^{-x^2/4t} + x\operatorname{erf}\frac{x}{2\sqrt{t}}\Big) + C_2 x$$

If we put $-C_2 = 2C_1 = 1$, we then have the solution

$$u(x,t) = \sqrt{t}e^{-x^2/4t} - x\operatorname{erfc}\frac{x}{2\sqrt{t}}$$

which satisfies

$$u(x,0) = 0, \quad \partial_x u(0,t) = -1$$

Appendix: The error function. The *error function* erf and the *complementary error function* are defined by

$$\operatorname{erf}\eta = \frac{2}{\sqrt{\pi}}\int_0^{\eta} e^{-\zeta^2}\,\mathrm{d}\zeta, \quad \operatorname{erfc}\eta = \frac{2}{\sqrt{\pi}}\int_{\eta}^{\infty} e^{-\zeta^2}\,\mathrm{d}\zeta.$$

Note that

$$\operatorname{erf}\eta + \operatorname{erfc}\eta = 1, \quad \operatorname{erf}0 = \operatorname{erfc}\infty = 0, \quad \operatorname{erf}\infty = \operatorname{erfc}0 = 1.$$