Suggested solution Mathematical modelling
Exercise 4, autumn 2005

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Exercise 1 – Roots of equations

We are going to find the roots of the equation

$$\epsilon x^3 + x - 1 = 0. \tag{1}$$

We first expand x as

$$x = x_0 + \epsilon x_1 + \epsilon^2 x_2 + \dots$$

By plotting the first function in (1) we realize that there is one real root, around $x = 1$ and two imaginary roots. We first consider balancing the terms x and -1. Inserting the expansion, doing some rearrangements and equating coefficients for the powers of ϵ yields

$$\begin{aligned} \epsilon^0 &: \ x_0 - 1 = 0 \\ \epsilon^1 &: \ x_0^3 + x_1 = 0 \\ \epsilon^2 &: \ 3x_0^2 x_1 + x_2 = 0. \end{aligned}$$

Solving these equations yields

$$x = 1 - \epsilon + 3\epsilon^2 + \dots.$$

We now balance the terms ϵx^3 and x. This gives a scaling

$$x = \frac{1}{\sqrt{\epsilon}} X.$$

Inserting this into the function yields

$$X^3 + X - \sqrt{\epsilon} = 0.$$

Letting $\epsilon = 0$ produces the uninteresting root $X = 0$ and the two roots $\pm i$, which leads to an expansion

$$X = \pm i + \epsilon^{1/2} X_1 + \epsilon X_2 + \dots$$

We once again proceed by equating powers of ϵ. This time the equations are

$$\begin{aligned} \epsilon^{1/2} &: \ -3X_1 + X_1 - 1 = 0 \\ \epsilon^1 &: \ -2X_2 \pm 3i X_1^2 = 0. \end{aligned}$$

The solution of these equations yields the following expressions for the two other roots:

$$x = \frac{1}{\sqrt{\epsilon}} \left(\pm i - \frac{\sqrt{\epsilon}}{2} \mp \frac{3i\epsilon}{8} + \dots \right)$$

Next, we are going to find the roots of the equation

$$\epsilon\, x \tan x - 1 = 0. \tag{2}$$

By looking at the plot of the function you should agree that

$$x = \left(n + \tfrac{1}{2}\right)\pi - \epsilon\, X$$

seems to be a good choice for a scaling. Inserting this into (2) yields

$$\epsilon\big((n+\tfrac{1}{2})\pi - \epsilon\, X\big)\frac{\sin\big((n+\tfrac{1}{2})\pi - \epsilon\, X\big)}{\cos\big((n+\tfrac{1}{2})\pi - \epsilon\, X\big)} = 1.$$

Since $\sin\big((n+\tfrac{1}{2})\pi - x\big) = -\cos x$ and $\cos\big((n+\tfrac{1}{2})\pi - x\big) = -\sin x$ this can be written as

$$\epsilon\big((n+\tfrac{1}{2})\pi - \epsilon\, X\big)\cos(\epsilon\, X) = \sin(\epsilon\, X).$$

We now expand X as

$$X = X_0 + \epsilon\, X_1 + \cdots$$

and use the Taylor expansion of the sine and cosine. After inserting these, we equate powers of ϵ. This yields

$$\begin{aligned} \epsilon^1 :&\ \left(n+\tfrac{1}{2}\right)\pi = X_0 \\ \epsilon^2 :&\ -X_0 = X_1 \end{aligned}$$

The solution of these equations yields

$$x = \left(n + \tfrac{1}{2}\right)\pi(1 - \epsilon + \epsilon^2 + \cdots).$$

Finally we are going to find the roots of the equation

$$x \tan x - \epsilon = 0 \tag{3}$$

By looking at a plot of the function, we realize that we have two roots around $x = 0$ and then a root around every whole multiple of π.

To find the first root use the scaling $x = \sqrt{\epsilon} X$. Inserting this into (3) yields

$$\sqrt{\epsilon} X \tan\left(\sqrt{\epsilon} X\right) - \epsilon = 0.$$

We now expand X as

$$X = X_0 + \epsilon X_1 + \epsilon^2 X_2 + \cdots .^1$$

Inserting this and equating powers we need to solve the equations

$$\begin{aligned} \epsilon^1 :&\ X_0^2 - 1 = 0 \\ \epsilon^2 :&\ 2X_0 X_1 + \frac{X_0^4}{3} = 0 \\ \epsilon^3 :&\ X_1^2 + 2X_0 X_2 + \frac{4X_0^3 X_1}{3} = 0. \end{aligned}$$

This results in the roots

$$\begin{aligned} x &= \sqrt{\epsilon}\left(1 - \frac{\epsilon}{6} + \frac{7\epsilon^2}{72} + \cdots\right) \\ x &= \sqrt{\epsilon}\left(-1 + \frac{\epsilon}{6} - \frac{7\epsilon^2}{72} + \cdots\right) \end{aligned}$$

To find the other roots we use the scaling $x = n\pi + \epsilon X$. Inserting this into (3) (remember that $\tan(n\pi + x) = \tan x$) we get

$$(n\pi + \epsilon X) \tan(\epsilon X) - \epsilon = 0.$$

We then expand X as

$$X = X_0 + \epsilon X_1 + \epsilon^2 X_2 + \cdots .$$

This time we end up with the three equations

$$\begin{aligned} \epsilon^1 :&\ n\pi X_0 - 1 = 0 \\ \epsilon^2 :&\ X_0^2 + n\pi X_1 = 0 \\ \epsilon^3 :&\ n\pi X_2 + 2X_0 X_1 + \frac{n\pi X_0^3}{3} = 0, \end{aligned}$$

which yields the solution

$$x = n\pi + \frac{\epsilon}{n\pi} - \frac{\epsilon^2}{(n\pi)^3} + \frac{2\epsilon^3}{(n\pi)^5} - \frac{\epsilon^3}{3(n\pi)^3} + \cdots .$$

[1] I told some of you to expand this as $X = X_0 + \sqrt{\epsilon} X_1 + \epsilon X_2$. It was the other way around, sorry!

Exercise 6 – Flagpoles again

We have been given the problem

$$\begin{gathered}\frac{\partial^2 y}{\partial t^2} + \alpha^4 \frac{\partial^4 y}{\partial x^4} = 0 \\ y_{xx} = y_{xxx} = 0 \quad \text{at } x = 1, \qquad y = \cos t, \quad y_x = 0 \quad \text{at } x = 0\end{gathered} \tag{4}$$

where $\alpha \gg 1$ and $\epsilon = \frac{1}{\alpha}$. We divide by α^4, and get the equation in the form

$$\begin{gathered}\epsilon^4 \frac{\partial^2 y}{\partial t^2} + \frac{\partial^4 y}{\partial x^4} = 0 \\ y_{xx} = y_{xxx} = 0 \quad \text{at } x = 1, \quad y = \cos t, y_x = 0 \quad \text{at } x = 0.\end{gathered} \tag{5}$$

We then let $y = y_0 + \epsilon^4 y_1 + \dots$ and insert this into (5). This yields

$$\epsilon^4 \left(\frac{\partial^2 y_0}{\partial t^2} + \epsilon^4 \frac{\partial^2 y_1}{\partial t^2} \right) + \frac{\partial^4 y_0}{\partial x^4} + \epsilon^4 \frac{\partial^4 y_1}{\partial x^4} = 0.$$

For ϵ^0 this yields the equation

$$\frac{\partial^4 y_0}{\partial x^4} = 0.$$

We then integrate four times and use the boundary conditions to find the constants of integration. This yields

$$y_0 = \cos t.$$

For the ϵ^4 term we get the equation

$$\frac{\partial^2 y_0}{\partial t^2} + \frac{\partial^4 y_1}{\partial x^4}.$$

We insert the known y_0 and integrate four times, with the result $y_1 = \frac{1}{24}(x^4 \cos t + ax^3 + bx^2 + cx + d)$. We use the boundary conditions $y_1 = y_{1x} = 0$ at $x = 0$ to conclude $c = d = 0$, and then the conditions $y_{1xx} = y_{1xxx} = 0$ at $x = 1$ to get $12\cos t + 6a + 2b = 0$ and $24\cos t + 6a = 0$. So $a = -4\cos t$ and $b = -6\cos t - 3a = 6\cos t$, and we have

$$y_1 = \frac{1}{24}(x^4 - 4x^3 + 6x^2)\cos t$$

and the final solution

$$y = \left(1 + \frac{1}{24}\epsilon^4 (x^4 - 4x^3 + 6x^2)\right)\cos t + \cdots.$$

Exercise 8 – The forced logistic equation

We are asked to explain why the equation

$$\frac{\mathrm{d}u}{\mathrm{d}t} = ku(1-u)$$

is a crude model for population dynamics. In general, the quantity

$$\frac{1}{u}\frac{\mathrm{d}u}{\mathrm{d}t}$$

is a measure of the *relative growth rate* in the population, measuring the rate of births, minus the rate of deaths, per individual and time unit. When the relative growth rate, we have the classical exponential growth (or decay, if the rate is negative).

In the present case

$$\frac{1}{u}\frac{\mathrm{d}u}{\mathrm{d}t} = k(1-u),$$

so the relative growth rate is near a constant k when $u \ll 1$, and it falls linearly to 0 when $u \to 1$. If $u > 1$, the population decreases. This is presumably due to some limited resource available to the population.

The term 1 is actually the term in the equation that corresponds to the size of the resource (the equation has been scaled in order to obtain the form given.

We now turn to the forced logistic equation

$$\frac{\mathrm{d}u}{\mathrm{d}t} = ku(1+\epsilon\cos t - u), \tag{6}$$

insert $u = 1 + \epsilon\, u_1(t) + \cdots$ and simplify. We get the ordinary differential equation

$$\frac{\mathrm{d}u_1}{\mathrm{d}t} + ku_1 = k\cos t,$$

which has the general solution

$$u_1 = \frac{k^2}{1+k^2}\cos t + \frac{k}{1+k^2}\sin t + c_1 e^{-kt}.$$

No matter the value of the constant c_1, it decays towards the periodic solution given by setting $c_1 = 0$.

We are then asked to show that we can solve (6) using a substitution $u = 1/v$. This yields $\mathrm{d}u/\mathrm{d}t = -v^{-2}\cdot \mathrm{d}v/\mathrm{d}t$, and after a bit of tidying we end up with a linear, nonhomogeneous equation with variable coefficents:

$$\frac{\mathrm{d}v}{\mathrm{d}t} + k(1+\epsilon\cos t)\,v = k.$$

This can be solved by multiplying with the integrating factor $e^{k(t+\epsilon\sin t)}$ and integrating, with the result

$$v = ke^{-k(t+\epsilon\sin t)}\int e^{k(t+\epsilon\sin t)}\,\mathrm{d}t.$$

That is perhaps not very illuminating! Our approximate formula

$$u = 1 + \epsilon\Big(\frac{k^2}{1+k^2}\cos t + \frac{k}{1+k^2}\sin t\Big) + O(\epsilon^2)$$

tells us much more, such as the size and phase of the oscillations of v.