Suggested solution Mathematical modelling
Exercise 9, autumn 2005

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This solution suggestion is written in a hurry and not carefully proofread. Also, it is not complete; but it should give you a good indication of the solution anyway.

Problem 1

(a) Written on conservation form, the model is

$$\rho_t + \big(\rho(1-\rho)\big)_x = 0,$$

indicating a flux $\rho(1-\rho)$ of runners. This corresponds to individual runners having a speed $1-\rho$ (this is obviously after some scaling has been performed). In the traffic model, cars rarely pass each other, so this really models individual cars. The runners will have different speeds, but if we interpret $1-\rho$ as the *average* speed of runners when the density is ρ, we still get the same model.

The 2π-periodicity simply corresponds to the track being circular (or oval really). The total track length has been scaled to 2π.

(b) The characteristic speed is $1-2\rho$, so the characteristic starting at $(\xi,0)$ has speed $1-2\rho_0-2\varepsilon\cos\xi$, and therefore its equation is $x=\xi+(1-2\rho_0-2\varepsilon\cos\xi)t$. In other words, the solution is implicitly given by

$$\rho(x,t)=\rho_0+\varepsilon\cos\xi, \qquad \xi-2\varepsilon t\cos\xi = x-(1+2\rho_0)t.$$

For given (x,t) one must solve the equation on the right for ξ, then insert this in the equation on the left. This works fine in principle so long as $\xi-2\varepsilon t\cos\xi$ is an increasing function of ξ, that is so long as $1+2\varepsilon t\sin\xi>0$ for all ξ. Obviously, this is fine so long as $2\varepsilon t<1$.

(c) (There should be a sketch here. I would have drawn one by hand, but don't have access to a scanner where I am at the moment. It takes too long to make a computer drawing, sorry.)

When $2\varepsilon t=1$ characteristics start to collide: From the above calculation this happens where $\sin\xi=-1$, that is $\xi=(2k+\frac{3}{2})\pi$. This characteristic has speed $1-2\rho_0$, so the shock collision happens at

$$(x,t)=\big((2k+\tfrac{3}{2}\pi)+(1-2\rho_0)/(2\varepsilon),2/\varepsilon\big) \qquad k=0,\pm1,\pm2,\dots$$

(d) We take the hint and look at characteristics starting at $x=\frac{3}{2}\pi\pm\theta$. These two characteristics have speeds $1-2\rho_0-2\varepsilon\cos(\frac{3}{2}\pi\pm\theta)=1-2\rho_0\mp2\varepsilon\sin\theta$. The difference in speeds is $4\varepsilon\sin\theta$, and the starting distance between the characteristics is 2θ, so they collide after a time $t=\theta/(\varepsilon\sin\theta)$. The position of this collision is $\frac{3}{2}\pi+(1-2\rho_0)t$. So if we guess that what we have just found is the shock, then that shock is moving with speed $1-2\rho_0$. Does it satisfy the Rankine–Hugoniot condition? The value of ρ on the two sides will be $\rho_0\pm\varepsilon\sin\theta$, so with $f(\rho)=\rho(1-\rho)=\rho-\rho^2$ we should find the shock speed

$$\frac{[f(\rho)]}{\rho}=\frac{\rho_+-\rho_+^2-\rho_-+\rho_-^2}{\rho_+-\rho_-}=1-(\rho_++\rho_-)=1-2\rho_0,$$

which fits the above finding just fine.

What to do without the hint? The Rankine–Hugoniot condition becomes a differential equation to be solved for the shock position, with left and right values of ρ given by the formulas above. Since these formulas only give *implicit* formulas for $\rho(x,t)$, this could become rather difficult.

Problem 2 The outer problem is $yy' = 1$, with general solution $\frac{1}{2}y^2 = x + A$.

It is not obvious at what end to apply the given boundary condition. We can try one, then the other, and use the one that works, or we can make an educated guess. One way to make such a guess is to transform this into a system:

$$y' = z, \quad \varepsilon z' = yz - 1.$$

This system is associated with a vector field $(z, (yz-1)/\varepsilon)$. When $0 < \varepsilon \ll 1$, then this vector field is very large except near the hyperbola $yz = 1$, which is therefore often called the *slow curve*.

On the part of the slow curve in the first quadrant, the vector field points to the right. But above the slow curve, the vector field has a large component pointing up, and below the slow curve, it points down. So when we move forward in "time" (think of x as time here), the slow curve is unstable, and the solution tends to move rapidly away from it. When we move backwards in "time", the solution moves rapidly towards the slow curve. All this indicates that a boundary layer should be at the right end of the interval.

So we apply the boundary condition $y(0) = 1$ to the outer solution and end up with

$$y_0 = \sqrt{2x+1}.$$

Note the value at the other end of the interval: $y_0(1) = \sqrt{3}$.

The *inner problem* for the boundary layer at $X = 1$ is given by setting $x = 1 - \varepsilon X$ and $y(x) = Y(X)$.[1] This leads to the equation $Y'' + YY' + \varepsilon = 0$ (note the change in sign because of the minus sign in $x = 1 - \varepsilon X$).

To lowest order we set $\varepsilon = 0$, so we get $Y'' + YY' = 0$, which can be integrated once to $Y' + \frac{1}{2}Y^2 = B$.

It will save us a bit of work if we can discover the value of B now: The lowest order matching condition becomes $Y(\infty) = y(1) = \sqrt{3}$. If Y is to have a limit at $X = \infty$ at all, then Y' must go to zero there.[2] So we must have $B = \frac{3}{2}$. The equation for Y is therefore $Y' = \frac{1}{2}(3 - Y^2)$. This is separable, with the general solution

$$\frac{1}{\sqrt{3}} \ln \frac{\sqrt{3}+Y}{\sqrt{3}-Y} = X + C.$$

It is time to use the boundary condition $y(1) = 0$, which becomes $Y(0) = 0$. So we insert $X = Y = 0$ above and get $C = 0$, and

$$Y = \sqrt{3}\frac{e^{\sqrt{3}X} - 1}{e^{\sqrt{3}X} + 1} = \sqrt{3}\tanh \tfrac{1}{2}\sqrt{3}X.$$

Altogether, a reasonable approximation should be "outer plus inner minus common part $\varepsilon\sqrt{3}$":

$$y = \sqrt{2x+1} + \sqrt{3}\tanh\Big(\sqrt{3}\,\frac{1-x}{2\varepsilon}\Big) - \sqrt{3}.$$

[1] I chose the minus sign because I like to have $X > 0$ in the inner region.

[2] This is not true for all functions, but it *is* true for functions satisfying a differential equation of the type we are considering here.