Suggested solution Mathematical modelling
Exercise 10, autumn 2005

Harald Hanche-Olsen

27. november 2005

Exercise 1 Clearly, the given equation implies that $\dot{N} < 0$ if $N > S$, while $\dot{N} > 0$ if $N < S$. This simple fact explains the term *sustainable* population: The population increases while it is smaller, and decreases when it is larger than, the sustainable population.

$\dot{N}/N$ is the *relative* growth rate: Growth per unit population and unit time. When $N \ll S$ then $\dot{N} \approx rN$, so the population grows exponentially with a time scale $1/r$. So $1/r$ is in fact a reasonable time scale. Picking S for the scale of N seems reasonable, since N will grow up to S with time.

These scalings ($N = SN'$ and $t = t'/r$) lead to the scaled equation

$$\frac{\dot{N}}{N} = 1 - N, \qquad \text{or equivalently:} \quad \dot{N} = N - N^2$$

where we have already dropped the primes.

Exercise 2 Clearly, the absolute harvest rates of krill and whales are $F_N N$ and $F_H H$ respectively: F_N and F_H could be called *relative* harvest rates: Proportions of the total population harvested per time unit.

With no whales eating krill ($H = 0$) and no harvest, the krill satisfy a logistic growth equation with sustainable population K.

The number of krill eaten by whales each year is of the form aHN: Clearly this means that each whale gets to eat an amount of krill proportional to the total krill population.

With no harvesting and a constant krill population, the whale population satisfies a logistic growth equation with sustainable population αN.

The text clearly implies that $1/q$ might be used for the time scale. We may also wish to scale the krill and whale populations, with K as the natural scale for N and hence αK as the natural scale for H.

If we insert $t = t'/r$, $N = KN'$ and $H = \alpha KH'$ in the given equations and then divide the first by r and the second by q and immediately drop the primes, we end up with

$$\begin{aligned} \epsilon \frac{\dot{N}}{N} &= 1 - N - \beta H - f_N, \\ \frac{\dot{H}}{H} &= 1 - \frac{H}{N} - f_H \end{aligned}$$

where we have put

$$\beta = a\alpha K, \quad f_N = F_N/r, \quad f_H = F_H/q.$$

(The latter two scalings seem to make sense, in that these dimensionless versions of harvest rates are expressed in terms of the natural reproduction rates of the two species.)

This system is clearly singular, in that its nature changes when $\epsilon \to 0$.

Equilibrium happens when

$$N + \beta H = 1 - f_N, \quad H = (1 - f_H)N.$$

Under the rather natural assumption that $0 \le f_N < 1$ and $0 \le f_H < 1$, these two equations describe a pair of lines that clearly intersect at a unique point in the first quadrant. (We do not bother with the simple computation.)

To investigate the stability of the system, rewrite it as

$$\begin{aligned}\dot{N} &= \frac{1}{\epsilon}\Big((1 - f_N)N - N^2 - \beta HN\Big),\\ \dot{H} &= (1 - f_H)H - \frac{H^2}{N}.\end{aligned}$$

To find its linearization at the equilibrium we just differentiate the righthand side wrt N and H, and build a matrix:

$$\begin{pmatrix} \dfrac{1 - fN - 2N - \beta H}{\epsilon} & -\dfrac{\beta N}{\epsilon} \\ \dfrac{H^2}{N^2} & 1 - f_H - 2\dfrac{H}{N} \end{pmatrix} = \begin{pmatrix} -\dfrac{N}{\epsilon} & -\beta N \\ \dfrac{H^2}{N^2} & f_h - 1 \end{pmatrix}$$

where we used the equilibrium equations to simplify the matrix. Given our assumptions the trace is clearly negative, and the determinant is positive – so the linearized system is *stable*. Hence, so is the equilibrium point in the nonlinear system.

With $\epsilon = 0$ the equations become

$$N + \beta H = 1 - f_N, \quad \frac{\dot{H}}{H} = 1 - \frac{H}{N} - f_H$$

which reduces to the single equation

$$\dot{H} = (1 - f_H)H - \frac{H^2}{\beta H - 1 + f_N}$$

for the whale population.

We expect the krill population to change rapidly towards the equilibrium value given by $N + \beta H = 1 - f_N$: Then the whale population changes more slowly according to the above differential equation while the krill population changes in the opposite direction.